

ON CATEGORY  $\mathcal{O}$  OVER TRIANGULAR GENERALIZED WEYL ALGEBRAS

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**ABSTRACT.** We analyze the BGG Category  $\mathcal{O}$  over a large class of generalized Weyl algebras (henceforth termed GWAs). Given such a “triangular” GWA for which Category  $\mathcal{O}$  decomposes into a direct sum of subcategories, we study in detail the homological properties of blocks with finitely many simples. As consequences, we show that the endomorphism algebra of a projective generator of such a block is quasi-hereditary, finite-dimensional, and graded Koszul. We also classify all tilting modules in the block, as well as all submodules of all projective and tilting modules. Finally, we present a novel connection between blocks of triangular GWAs and Young tableaux, which provides a combinatorial interpretation of morphisms and extensions between objects of the block.

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## 1. INTRODUCTION AND MAIN RESULTS

Generalized Weyl Algebras (GWAs) are an important and well-studied class of algebras in the literature. There is much recent activity on the study of GWAs, including their existence and consistency, structure, and representation theory, as well as of special sub-families of GWAs. The present paper provides a contribution to this area.

Recall [1] that a GWA is generated over a ring  $R$  (equipped with a ring automorphism  $\theta : R \rightarrow R$ ) by two elements  $u, d$  with the relations:  $ur = \theta(r)u, rd = d\theta(r)$  for all  $r \in R$ , and  $ud = \theta(du) \in Z(R)$ . We focus on the case when  $R = H[du] = H[ud]$  for a commutative  $\mathbb{F}$ -algebra  $H$  over a field  $\mathbb{F}$ ; in the present paper, (generalizations of) such algebras will be termed *triangular GWAs*. These algebras enjoy several desirable properties, including a triangular decomposition and an appropriate theory of weights. This allows the introduction and study of the Bernstein-Gelfand-Gelfand (BGG) Category  $\mathcal{O}$  over triangular GWAs. Our goal in this paper is to show that a large amount of homological information about Category  $\mathcal{O}$  can be obtained in a uniform manner for all triangular GWAs. Specifically, given a weight  $\lambda$  of  $H$ , we study the endomorphism algebra  $\mathbf{A}_{[\lambda]} = \text{End}_{\mathcal{O}}(\mathbf{P}_{[\lambda]})^{op}$  of a specific projective generator  $\mathbf{P}_{[\lambda]}$  of the corresponding block  $\mathcal{O}[\lambda]$  of  $\mathcal{O}$ . As a first step, a general treatment of Category  $\mathcal{O}$  can be used to show that when the block has finitely many simple objects, the algebra  $\mathbf{A}_{[\lambda]}$  is  $\mathbb{Z}_+$ -graded, associative, finite-dimensional, and quasi-hereditary.

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In this paper we compute all Ext-groups for pairs of simple modules, Verma modules, or (quotients of) projective modules, as well as the Ext-groups between these modules, for a general triangular GWA. Our results yield many desirable homological consequences for blocks of triangular GWAs. First, we provide a presentation for the algebras  $\mathbf{A}_{[\lambda]}$  and show that the isomorphism class of the algebra depends only on the (finite) number of simple objects in the block. In particular, this shows that all (finite) blocks of triangular GWAs with equal numbers of simple objects are Morita equivalent.

Second, we prove that the algebras  $\mathbf{A}_{[\lambda]}$  are Koszul. Koszulity is an important structural property for  $\mathbb{Z}_+$ -graded, quadratic algebras and has several desirable homological consequences; see e.g. [2, 12, 35, 36] for more on Koszulity and its generalizations.

An additional consequence is a complete description of all tilting modules in blocks of Category  $\mathcal{O}$ , as well as an enumeration of all submodules of projective or tilting objects in a block. Specifically, we show that each such submodule is indecomposable and has a Verma flag.

A fourth consequence is an interesting and novel connection to Young-type tableaux, which to our knowledge has not been explored in the literature. These tableaux satisfy combinatorial counterparts of our homological results, as we explain in this paper. In other words, blocks of  $\mathcal{O}$  categorify Young tableaux.

Finally, the complete and explicit descriptions afforded by our computations make it possible to apply the comprehensive homological machinery developed by Cline, Parshall, and Scott in their broad program for highest weight categories. For instance, we show as a corollary of our results that the blocks of Category  $\mathcal{O}$  satisfy the *Strong Kazhdan-Lusztig condition (SKL)* as in [14].

**1.1. Triangular GWAs.** We now develop the notation required to present the main results later in this section. We begin by introducing the main object of study in the present paper. For this paper we fix an arbitrary ground field  $\mathbb{F}$ ; thus,  $\dim$  henceforth denotes  $\dim_{\mathbb{F}}$ . Also, let  $\mathbb{Z}_+$  denote the set of non-negative integers.

**Definition 1.1.** Suppose  $H$  is a commutative  $\mathbb{F}$ -algebra with an  $\mathbb{F}$ -algebra automorphism  $\theta : H \rightarrow H$ , and elements  $z_0 \in H, z_1 \in H^\times$ . The *triangular Generalized Weyl Algebra (triangular GWA)* associated to this data is defined to be the  $\mathbb{F}$ -algebra

$$\mathcal{W}(H, \theta, z_0, z_1) := H\langle d, u \rangle / (uh = \theta(h)u, \quad hd = d\theta(h), \quad ud = z_0 + dz_1u \quad \forall h \in H). \quad (1.2)$$

Triangular GWAs are the focus of a concerted research effort in the literature. A large class of triangular GWAs that has been the focus of much recent research consists of *down-up algebras*. These are a family of generalized Weyl algebras that occur in several different settings, including representation theory, mathematical physics, Hopf algebras, ring theory, and combinatorics. See [23, 30, 32, 37, 39, 41, 44] for these and other motivations. It turns out that the algebras in the above references have certain common structure and properties. For instance, they contain elements  $d$  and  $u$  that should be thought of as “down” (lowering) and “up” (raising) operators. In order to systematically study their behavior, Benkart and Roby [3] defined *down-up algebras* and initiated their study. Since then, down-up algebras and their variants have been the focus of tremendous interest - to name a few references, see [10, 11, 25, 29, 31, 33, 38, 46]. Other examples of down-up algebras have been studied by Woronowicz [45], as well as Kac in the comprehensive work [27] on Lie superalgebras. We remark that down-up algebras are a sub-family of triangular GWAs with  $H = \mathbb{F}[h]$ , a polynomial algebra; see [28, Section 8] for more details.

Simultaneously, another area of much recent interest is the study of various “quantum” and Hopf-like algebras. These “quantum” variants are generated by  $u, d$  over the group ring  $\mathbb{F}[\Gamma]$  for some group  $\Gamma$ . As above, examples have arisen from a variety of settings, including Kleinian singularities and quantum groups. See [15, 21, 22, 42, 43, 47] for more references. As above, all of these “quantum” down-up algebras are triangular GWAs with  $H = \mathbb{F}[K^{\pm 1}]$ , a group algebra -

see the discussion in [28, Section 8]. It is also shown in *loc. cit.* that the down-up algebras in the former, “classical” family, admit quantizations that belong to the latter, “quantum” family.

Both classical and quantum down-up algebras are special cases of *ambiskew polynomial rings*, which are the class of triangular GWAs where  $z_1 \in \mathbb{F}^\times$ . Ambiskew polynomial rings are the focus of recent and continuing interest [9, 19, 24, 26]. Generalized Weyl algebras can also arise from other constructions. For instance as explained in [28, Section 9], continuous Hecke algebras of  $GL(1)$  and  $\mathbb{C} \oplus \mathbb{C}^*$  (see [17]) are generalized Weyl algebras. Thus our goal in the present paper is to prove results for general triangular GWAs, addressing uniformly all of the above examples.

**1.2. Category  $\mathcal{O}$ .** In order to state the main results in this paper, we now introduce a sequence  $\tilde{z}_n$  of distinguished elements in a triangular GWA (more precisely, in its “Cartan subalgebra”  $H$ ). We also set further notation.

**Definition 1.3.** Fix an  $\mathbb{F}$ -algebra  $H$ , elements  $z_0 \in H$  and  $z_1 \in H^\times$ , and an  $\mathbb{F}$ -algebra automorphism  $\theta : H \rightarrow H$ . Now let  $A := \mathcal{W}(H, \theta, z_0, z_1)$  be the algebra defined as in Equation (1.2).

(1) Given an integer  $n \geq 1$ , define

$$z'_n := \prod_{i=0}^{n-1} \theta^i(z_1), \quad z'_0 := 1, \quad \tilde{z}_n := \sum_{i=0}^{n-1} \theta^i(z_0 z'_{n-1-i}), \quad \tilde{z}_0 := 0, \quad \tilde{z}_{-n} := \theta^{-n}(\tilde{z}_n). \quad (1.4)$$

(2) Define a *character* or *weight* of  $H$  to be an  $\mathbb{F}$ -algebra map  $\lambda : H \rightarrow \mathbb{F}$ , and denote the set of weights of  $H$  by  $\hat{H} := \text{Hom}_{\mathbb{F}\text{-alg}}(H, \mathbb{F})$ . Now given a weight  $\lambda \in \hat{H}$ , define

$$[\lambda] := \{\lambda \circ \theta^n : n \in \mathbb{Z}, \lambda(\tilde{z}_n) = 0\}, \quad (1.5)$$

$$\hat{H}^{free} := \{\lambda \in \hat{H} : \forall n \in \mathbb{Z} \setminus \{0\}, \exists h \in H \text{ with } \lambda(h) \neq (\lambda \circ \theta^n)(h)\}.$$

(3) Given an  $H$ -module  $M$  and  $\lambda \in \hat{H}$ , the  $\lambda$ -*weight space* of  $M$  is  $M_\lambda := \{m \in M : \ker(\lambda)m = 0\}$ . Now define  $\text{wt } M := \{\lambda \in \hat{H} : M_\lambda \neq 0\}$ . We say that  $M$  is a *weight module* over  $H$  if  $M = \bigoplus_{\lambda \in \hat{H}} M_\lambda$ .

(4) Define the *BGG Category  $\mathcal{O}$*  over  $A$  to be the full subcategory of all finitely generated  $H$ -weight  $A$ -modules, with finite-dimensional  $H$ -weight spaces and a locally finite action of  $u$ .

(5) We show in Remark 2.9 below that Category  $\mathcal{O}$  contains pairwise non-isomorphic simple objects  $L(\lambda)$  for all  $\lambda \in \hat{H}^{free}$ . Now given a subset  $T \subset \hat{H}^{free}$ , define  $\mathcal{O}(T)$  to be the full subcategory of all objects in  $\mathcal{O}$ , each of whose Jordan-Holder subquotients is  $L(\lambda)$  for some  $\lambda \in T$ . Also let  $\mathcal{O}_{\mathbb{N}}$  denote the full subcategory of all finite length objects in  $\mathcal{O}$ , and define  $\mathcal{O}_{\mathbb{N}}(T) := \mathcal{O}_{\mathbb{N}} \cap \mathcal{O}(T)$ . If  $T = [\lambda]$ , denote  $\mathcal{O}[\lambda] := \mathcal{O}([\lambda])$  and  $\mathcal{O}_{\mathbb{N}}[\lambda] := \mathcal{O}_{\mathbb{N}}([\lambda])$ .

It is then clear that  $\mathbb{Z}$  acts on the set of weights  $\lambda : H \rightarrow \mathbb{F}$  (and this action is free on the subset  $\hat{H}^{free}$ ), via:  $n * \lambda := \lambda \circ \theta^{-n}$ . This yields a partial order on  $\hat{H}$ , via:  $\lambda < n * \lambda$  for all  $n > 0$  and  $\lambda \in \hat{H}^{free}$ . Throughout this paper we will use the following (slight) abuse of notation without further reference:  $\lambda - \mu = n \in \mathbb{Z}$  if  $n * \mu = \lambda$  for  $\lambda, \mu \in \hat{H}^{free}$ . The following identity is also useful in this setting, and easily verified:

$$\tilde{z}_{m+n} = \tilde{z}_n \theta^n(z'_m) + \theta^n(\tilde{z}_m), \quad \forall n, m \geq 0. \quad (1.6)$$

From (1.6), it follows easily that  $[\lambda] = [\mu]$  whenever  $\mu \in [\lambda]$ .

**Remark 1.7.** We briefly elaborate on the set  $[\lambda]$ . It turns out that  $\mu \in [\lambda]$  if and only if either  $[M(\lambda) : L(\mu)] > 0$  or  $[M(\mu) : L(\lambda)] = 0$ . In other words, if  $\lambda > \mu$ , then  $u \cdot d^{\lambda-\mu} m_\lambda = 0$ . This is akin to a maximal/primitive vector for the positive nilpotent Lie subalgebra  $\mathfrak{n}^+$ , for a semisimple Lie algebra. Another way to view  $[\lambda]$ , if  $z_1 = 1$  and  $z_0 \in \text{im}(\text{id}_H - \theta)$ , is as follows: under these assumptions  $A$  has a central Casimir operator  $\Omega$  (see [28, Section 8]); then  $[\lambda]$  is precisely the set of weights  $\mu \in \mathbb{Z} * \lambda$  for which the central characters on the Verma modules  $M(\mu), M(\lambda)$  coincide.

**1.3. Main results.** To state our main results, we require the notion of a *Koszul algebra*, which is a useful homological property in the study of quadratic algebras [35], prominent in representation theory.

**Definition 1.8** ([2, Definition 1.1.2]). A ring  $A$  is said to be *Koszul* if it satisfies the following conditions:

- (1)  $A$  is a  $\mathbb{Z}_+$ -graded ring, with  $A_0$  a semisimple subring.
- (2) The graded left  $A$ -module  $A_0$  admits a graded projective resolution

$$\cdots \rightarrow P^2 \rightarrow P^1 \rightarrow P^0 \rightarrow A_0$$

such that  $P^i$  is generated by its degree  $i$  component, i.e.,  $P^i = AP_i^i \forall i \geq 0$ .

Also define, for a  $\mathbb{Z}_+$ -graded ring  $A = \bigoplus_{j \geq 0} A_j$ , its *homological dual*  $E(A) := \text{Ext}_A^\bullet(A_0, A_0)$ .

Similarly, one also defines for a quadratic  $\mathbb{F}$ -algebra  $A = T(V)/(Q)$  (with  $Q \subset V \otimes V$ ), its *quadratic dual*  $A^! := T(V^*)/(Q^\perp)$ . Then the following properties of Koszul algebras are well-known.

**Theorem 1.9** ([2, Section 2]). *Suppose  $A$  is a finite-dimensional Koszul algebra over a field  $\mathbb{F}$ . Then  $A$  is quadratic. Moreover,  $E(A)$  is also Koszul, and is isomorphic as an  $\mathbb{Z}_+$ -graded  $\mathbb{F}$ -algebra to  $(A^!)^{op}$ , whence  $E(E(A)) \cong A$ .*

Using the above notation, it is possible to state the first main result of this paper.

**Theorem A.** *Suppose  $\mathcal{W}(H, \theta, z_0, z_1)$  is a triangular GWA, for which  $\widehat{H}^{free}$  is non-empty and Category  $\mathcal{O}$  is finite length. Suppose  $[\lambda]$  is finite for some  $\lambda \in \widehat{H}^{free}$ .*

- (1) *Let  $\{L_i : 1 \leq i \leq n = |[\lambda]|\}$  be the set of simple objects in  $\mathcal{O}[\lambda]$ , and  $P_i$  be the projective cover of  $L_i$  in the block. Then  $\mathcal{O}[\lambda]$  is equivalent to  $\mathbf{A}_{[\lambda]} - \text{Mod}$ , where  $\mathbf{A}_{[\lambda]} = \text{End}_{\mathcal{O}}(\bigoplus_i P_i)^{op}$  is a  $\mathbb{Z}_+$ -graded  $\mathbb{F}$ -algebra of dimension  $1^2 + \cdots + n^2$ , which is quasi-hereditary and Koszul.*
- (2) *The Ext-quiver of  $\mathbf{A}_{[\lambda]}$  is the double  $\overline{A_n}$  of the  $A_n$ -quiver*

$$[1] \rightarrow [2] \rightarrow \cdots \rightarrow [n].$$

*Label the arrows as  $\gamma_i : [i+1] \rightarrow [i]$  and  $\delta_i : [i] \rightarrow [i+1]$ . Then  $\mathbf{A}_{[\lambda]}^{op}$  is isomorphic to the path algebra of the quiver  $\overline{A_n}$  with relations*

$$\delta_i \circ \gamma_i = \gamma_{i+1} \circ \delta_{i+1} \quad \forall 0 < i < n-1, \quad \delta_{n-1} \circ \gamma_{n-1} = 0. \quad (1.10)$$

Thus at its heart, Category  $\mathcal{O}$  over every triangular GWA (with commutative  $H$ ) is governed by a distinguished family of finite-dimensional Koszul algebras  $\mathbf{A}_{[\lambda]}$ , which may be denoted by  $\mathcal{A}_n$  to denote their dependence only on the integer  $n = |[\lambda]| \geq 1$ . In particular, all finite blocks of Category  $\mathcal{O}$  over any triangular GWA, having exactly  $n$  simple objects, are Morita equivalent to finite-dimensional  $\mathcal{A}_n$ -modules. We also remark that the algebras  $\mathcal{A}_n$  have connections to other settings in representation theory; see Remark 5.2 for more details.

Note that Theorem A holds for a very large class of generalized Weyl algebras. For instance, it has the following consequence that applies to a large class of algebras described above in this section.

**Corollary 1.11.** *Suppose  $\text{char } \mathbb{F} = 0$ , and  $A = \mathcal{W}(\mathbb{F}[h], \theta, s^{-1}f(h), s^{-1})$  is a generalized down-up algebra with  $r = 1$ ,  $\gamma \in \mathbb{F}^\times$ , and  $0 \neq f \in \mathbb{F}[h]$ . Also suppose that  $s = 1$  or  $s$  is not a root of unity. Then:*

- (1) *Category  $\mathcal{O}$  over  $A$  has a block decomposition into summands  $\mathcal{O}[\lambda]$ .*
- (2) *Every block  $\mathcal{O}[\lambda]$  contains only finitely many non-isomorphic simple objects.*
- (3) *For each block  $\mathcal{O}[\lambda]$ , the corresponding algebra  $\mathbf{A}_{[\lambda]}$  is Koszul.*

Similar results also hold for “quantum” analogues of such algebras (mentioned above). Namely, suppose

$$\text{char } \mathbb{F} = 0, \quad \Gamma = \langle K, K^{-1} \rangle \cong \mathbb{Z}, \quad \theta(K) = qK, \quad z_1 \in \mathbb{F}^\times,$$

with  $q \in \mathbb{F}^\times$  not a root of unity. Also suppose that  $z_0 \in \mathbb{F}[K^{\pm 1}]$  is not of the form  $bK^n$  for any  $b \in \mathbb{F}, n \in \mathbb{Z}$ . Then the three assertions above (in this corollary) hold for  $A = \mathcal{W}(H, \theta, z_0, z_1)$ .

In particular, Theorem A holds for Smith’s family of algebras [39] with  $[x, y] = f(h) \neq 0 = \text{char } \mathbb{F}$ , as well as for the “quantized version” of Smith’s algebras studied by Ji et al. [22] and Tang [42], as long as  $q$  is not a root of unity and  $z_0 \notin \bigcup_{n \in \mathbb{Z}} \mathbb{F}K^n$ .

We also remark that Theorem A can be proved for an even larger class of algebras with triangular decomposition. See Remark 5.1.

The heart of the proof of Theorem A involves homological calculations in Category  $\mathcal{O}$  over a triangular GWA. This leads to our next main result.

**Theorem B.** (Setting as in Theorem A.) Suppose  $[\lambda] = \{\lambda_1 < \lambda_2 < \dots < \lambda_n\} \subset \widehat{H}^{\text{free}}$ . Then for all  $1 \leq i, j \leq n$  and  $l > 0$ ,

$$\text{Ext}_{\mathcal{O}}^l(L(\lambda_i), L(\lambda_j)) = \begin{cases} \mathbb{F}, & \text{if } |i - j| = l = 0; \\ \mathbb{F}, & \text{if } |i - j| = l = 1; \\ \mathbb{F}, & \text{if } i = j \neq 1 \text{ and } l = 2; \\ 0, & \text{otherwise.} \end{cases} \quad (1.12)$$

Now define for  $1 \leq i \leq n$ :

$$L_i := L(\lambda_i), \quad M_i := M(\lambda_i), \quad P_i := P(\lambda_i), \quad P_{n+1} := 0 =: L_0, \quad (1.13)$$

where  $P(\lambda)$  denotes the projective cover of  $L(\lambda)$  in  $\mathcal{O}[\lambda]$ , and  $M(\lambda) = A/(A \cdot u + A \cdot \ker \lambda)$  is the “Verma module” of highest weight  $\lambda$ . Then for all  $1 \leq i \leq n$ ,  $M_i$  has a finite filtration

$$M_i \supset M_{i-1} \supset \dots \supset M_1 \supset 0 =: M_0,$$

with successive subquotients  $L_j$  for  $1 \leq j \leq i$ . Similarly, every  $P_i$  has a “Verma flag”

$$P_i \supset P_{i+1} \supset \dots \supset P_n \supset 0,$$

with successive subquotients  $M_j$  for  $i \leq j \leq n$ . Moreover, for all  $1 \leq j < k \leq n+1$  and  $0 \leq s < r \leq n$ , defining  $\mathbf{1}(E)$  for a mathematical condition  $E$  to be 1 when the condition  $E$  holds, and 0 otherwise, we have:

$$\begin{aligned} \dim \text{Ext}_{\mathcal{O}}^l(M_r, P_j/P_k) &= \delta_{l,0} \mathbf{1}(r < k) + \delta_{l,1} \mathbf{1}(r < j), \\ \dim \text{Ext}_{\mathcal{O}}^l(P_j/P_k, M_r/M_s) &= \delta_{l,0} \mathbf{1}(s < j \leq r) + \delta_{l,1} \mathbf{1}(s < k \leq r). \end{aligned} \quad (1.14)$$

Theorem B summarizes important homological information in the block  $\mathcal{O}[\lambda]$ . For instance, in the special case  $k = j+1$ , Theorem B computes all Ext-groups between Verma modules and highest weight modules.

Recall that the definition of Koszulity involves the Ext-algebra  $E(\mathbf{A}_{[\lambda]})$ . Our next main result involves understanding the structure of  $E(\widetilde{\mathbf{A}}_{[\lambda]})$ , where  $\widetilde{\mathbf{A}}_{[\lambda]}$  is the larger algebra given by

$$\widetilde{\mathbf{A}}_{[\lambda]}^{\text{op}} := \text{End}_{\mathcal{O}} \bigoplus_{1 \leq r < s \leq n+1} P_r/P_s. \quad (1.15)$$

In turn, this enables a detailed analysis of projective objects in the highest weight category  $\mathcal{O}[\lambda]$ , as well as a complete classification of indecomposable injective and tilting modules (i.e., modules that have both a Verma flag as well as a dual Verma flag).

**Theorem C.** Setting as in Theorems A and B.

- (1) Fix integers  $1 \leq j < k \leq n+1$  and  $1 \leq r < s \leq n+1$ . Then,
- $$\dim \operatorname{Ext}_{\mathcal{O}}^l(P_r/P_s, P_j/P_k) = \begin{cases} \mathbf{1}(r < k) \min(s-r, k-r, k-j), & \text{if } l = 0; \\ \mathbf{1}(r \leq j) \mathbf{1}(s \leq k) (\min(0, j-s) + \min(s-r, k-j)), & \text{if } l = 1; \\ 0, & \text{otherwise.} \end{cases} \quad (1.16)$$
- (2) Given  $1 \leq r < s \leq n+1$ , there exists a bijection between the submodules of  $P_r/P_s$ , and strictly decreasing sequences of integers  $s-1 \geq m_l > m_{l-1} > \cdots > m_1 \geq 1$ , for some  $0 \leq l \leq s-r$ . Every such submodule is indecomposable and has a Verma flag, and the number of these submodules is  $\sum_{l=0}^{s-r} \binom{s-1}{l}$ .
- (3) The partial/indecomposable tilting modules in the block  $\mathcal{O}[\lambda]$  are precisely  $T_k := P_1/P_{k+1}$  for  $1 \leq k \leq n$ . Each of these modules is self-dual. In particular, the injective hull in the block  $\mathcal{O}[\lambda]$  of the simple module  $L_k$  is equal to  $F(P_k) \cong T_n/T_{k-1}$ , where we set  $T_0 := 0$ .

**Remark 1.17.** The condition that  $[\lambda] \subset \widehat{H}^{free}$  is a natural one to assume. In the special case of  $\mathcal{W}(H, \theta, z_0, z_1) = U(\mathfrak{sl}_2)$ , the condition amounts to requiring that  $\mathbb{F}$  has characteristic zero, while for  $\mathcal{W}(H, \theta, z_0, z_1) = U_q(\mathfrak{sl}_2)$ , the condition amounts to  $q$  not being a root of unity. Thus, this condition affords a “clean” picture in the case of a general triangular GWA, and allows us to focus on the technical issues of Koszulity and the structure of  $\mathbf{A}_{[\lambda]}, \widetilde{\mathbf{A}}_{[\lambda]}$ .

Observe that our main results do not make any assumption on the ground field (other than  $\widehat{H}^{free}$  being non-empty, which can entail  $\operatorname{char} \mathbb{F} = 0$ ). In particular, we do not require  $\mathbb{F}$  to be algebraically closed, as is the case in the literature when methods involving Gabriel’s theorem are used, to discuss the structure of basic, finite-dimensional, Koszul algebras. In this paper we do not use Gabriel’s result, but rather, rely on the comprehensive homological information that we derive about the algebras  $\mathbf{A}_{[\lambda]}$  and  $\widetilde{\mathbf{A}}_{[\lambda]}$  from Theorems B and C. Thus, we will first prove Theorems B and C, and then use these results to show the Koszulity and structure of the algebra  $\mathbf{A}_{[\lambda]}$  in Theorem A.

Finally, a novel feature of this paper involves introducing an appropriate combinatorial category of Young tableaux. This is carried out in Section 6, where we provide strong and novel homological connections between this category and all finite blocks  $\mathcal{O}[\lambda]$  for an arbitrary triangular GWA.

**Organization of the paper.** The remainder of this paper is organized as follows. In Section 2 we recall the standard approach for developing a theory of Category  $\mathcal{O}$  over a triangular GWA, leading up to the block decomposition of  $\mathcal{O}$  into highest weight categories. In Section 3 we prove a projective resolution of any simple module in  $\mathcal{O}[\lambda]$  and also prove Theorem B. Next, in Section 4 we study maps between the modules  $P_r/P_s$ , i.e. the algebra  $\widetilde{\mathbf{A}}_{[\lambda]}$ . Using this we classify all tilting modules, projective modules, and their submodules. This helps in proving Theorem C, and in Section 5, Theorem A. Finally, in Section 6 we define and study sub-triangular Young tableaux (STYT), and their many homological connections to the block  $\mathcal{O}[\lambda]$ .

## 2. PBW DECOMPOSITION AND THE BERNSTEIN-GELFAND-GELFAND CATEGORY

In this section, we list certain basic properties of triangular GWAs as well as Category  $\mathcal{O}$  over them. These properties will be used in proving our main results in the subsequent sections of the paper.

**2.1. PBW property.** We begin with a few preliminary observations on triangular GWAs. The results in this subsection are not hard to show, and we omit their proofs as they are relatively straightforward computations. The first observation is that if  $z_1$  is invertible in  $H$ , then the

triangular GWA  $\mathcal{W}(H, \theta, z_0, z_1)$  is in fact a generalized Weyl algebra over  $H[ud]$ , with  $\theta$  extended to  $H[ud]$  via:  $\theta(ud) = udz_1 + \theta(z_0)$ . This is made more precise in the following result.

**Lemma 2.1.** *Suppose  $H$  is an  $\mathbb{F}$ -algebra with automorphism  $\theta$  and  $z_0, z_1 \in H$ . Define  $A = \mathcal{W}(H, \theta, z_0, z_1)$  as in Equation (1.2).*

- (1) *Then  $ud, du$  commute with all of  $H$ . Moreover,  $H\langle ud \rangle = H\langle du \rangle$ .*
- (2)  *$du$  and  $ud$  are simultaneously algebraic or simultaneously transcendental over  $H$  (in  $A$ ).*
- (3) *If  $du$  is transcendental over  $H$  (in  $A$ ), then the following are equivalent:*
  - (a)  *$\theta$  extends to an  $\mathbb{F}$ -algebra automorphism of  $H\langle du \rangle = H[du] = H[ud]$ , and  $A$  is a Generalized Weyl Algebra of degree 1 over  $H[du]$ , with  $\theta(du) = ud = z_0 + dz_1u$ .*
  - (b)  *$z_1 \in H^\times$  is a unit in  $H$ .*

We next discuss a useful characterization of the transcendence of the elements  $du, ud$  over  $H$  in a triangular GWA. This characterization, called the *PBW property*, allows us to work with a distinguished  $\mathbb{F}$ -basis, and is explained as follows. A triangular GWA  $\mathcal{W}(H, \theta, z_0, z_1)$  is equipped with a  $\mathbb{Z}_+$ -filtration that assigns degree 0 to  $H$  and degree 1 to  $d, u$ . The associated graded algebra is the (possibly non-commutative) algebra  $\mathcal{W}(H, \theta, 0, z_1)$ . A natural question is to classify all of the flat – or *PBW* – deformations  $\mathcal{W}(H, \theta, z_0, z_1)$ . Recall that flat deformations can be characterized in terms of Ore extensions  $S[X; \sigma, \delta]$ , where  $\sigma$  is an algebra automorphism of the  $\mathbb{F}$ -algebra  $S$ , and  $\delta$  is a  $\sigma$ -derivation of  $S$ . Now note that  $H$  and  $u$  generate a semidirect product algebra  $H \ltimes \mathbb{F}[u]$ . Then the following result is not hard to show, and is used without reference throughout the remainder of the paper.

**Theorem 2.2** (PBW property). *Suppose  $H$  is an  $\mathbb{F}$ -algebra with automorphism  $\theta$  and  $z_0, z_1 \in H$ . Define  $A = \mathcal{W}(H, \theta, z_0, z_1)$  as in Equation (1.2). Then the following are equivalent:*

- (1)  *$\mathcal{W}(H, \theta, z_0, z_1)$  is a flat deformation of  $\mathcal{W}(H, \theta, 0, z_1)$ . (This is called the “PBW property”).*
- (2)  *$z_0, z_1$  are central in  $H$ .*
- (3) *The maps  $\sigma, \delta : H \ltimes \mathbb{F}[u] \rightarrow H \ltimes \mathbb{F}[u]$  given by*

$$\sigma(u) = z_1u, \quad \sigma|_H \equiv \theta, \quad \delta(u) = z_0, \quad \delta|_H \equiv 0$$

*are indeed an algebra automorphism and a  $\sigma$ -derivation respectively.*

*In particular,  $\mathcal{W}(H, \theta, z_0, z_1)$  is an Ore extension if these (equivalent) conditions hold:*

$$\mathcal{W}(H, \theta, z_0, z_1) = H[u; \theta^{-1}, 0][d; \sigma, \delta]. \quad (2.3)$$

*If, moreover,  $z_1$  is not a zero-divisor in  $H$ , then these conditions are equivalent to:*

- (4)  *$ud, du$  are transcendental over  $H$ .*

Note that such a deformation would have a “PBW”  $\mathbb{F}$ -basis  $\{d^r h_i u^s : 0 \leq r, s \in \mathbb{Z}, i \in I\}$ , where  $\{h_i : i \in I\}$  runs over an  $\mathbb{F}$ -basis of  $H$ . In [28, Section 8], it was explored if the aforementioned examples of triangular GWAs satisfied the assumptions of Theorems 2.2 and A.

In the proof of (1)  $\implies$  (4) in Theorem 2.2, certain computations are used that are also needed later in this paper. We now state these computations for future use.

**Proposition 2.4.** *Suppose  $H$  is an  $\mathbb{F}$ -algebra, with automorphism  $\theta$  and  $z_0, z_1 \in Z(H)$  central. Define  $A = \mathcal{W}(H, \theta, z_0, z_1)$  as in Equation (1.2).*

- (1) *The centralizers in  $H$  of  $u$  and  $d$  coincide:  $Z_H(u) = Z_H(d) = \ker(\text{id}_H - \theta)$ .*

(2) For all  $h, h_1, \dots, h_n \in H$  and integers  $0 \leq m \leq n$ ,

$$\begin{aligned} u \cdot d^m h &= d^n \theta(h) z'_n u + d^{n-1} \tilde{z}_n h, \\ \prod_{i=1}^n (dh_i u) &\in d^n \theta^{n-1}(h_1 \cdots h_n) \prod_{i=0}^{n-1} \theta^i(z'_{n-1-i}) u^n + \sum_{i=1}^{n-1} d^i \cdot H \cdot u^i, \\ u^m d^n &\in d^{n-m} \cdot \prod_{j=n-m}^{n-1} \tilde{z}_{j+1} + \mathcal{W}(H, \theta, z_0, z_1) \cdot u. \end{aligned} \quad (2.5)$$

(3) For all  $i, j, k, l \in \mathbb{Z}_+$  and  $h, h' \in H$ ,  $d^i h u^j \cdot d^k h' u^l \in \sum_{t=0}^{\min(j,k)} d^{i+k-t} \cdot H \cdot u^{j+l-t}$ .

The proofs of these statements are standard and are hence omitted.

In proving Theorem B, we require one further preliminary result in  $\mathcal{W}(H, \theta, z_0, z_1)$ .

**Proposition 2.6.** *Suppose  $A = \mathcal{W}(H, \theta, z_0, z_1)$  is a triangular GWA. Consider the grading on  $A$  with  $\deg u = 1, \deg d = -1, \deg H = 0$ . Then for all  $n \in \mathbb{Z}$ ,  $A[n]$  is isomorphic to  $H[du]$  as a left  $H[du]$ -module.*

*Proof.* It follows from Theorem 2.2 that the  $n$ th graded component  $A[n]$  of  $A$  is spanned by  $d^m H u^{m+n}$  for all  $m \geq \max(0, -n)$ . It follows from the PBW Theorem 2.2 that the result for  $n \geq 0$  reduces to that for  $n = 0$ . It thus suffices to show the result for  $n \leq 0$ .

First suppose that  $n = 0$ , and consider the identity map between the filtered vector spaces  $: H[du] \rightarrow A[0]$ , where the filtration is according to the length of the monomials in  $d, u$ . By the second of the equations (2.5) (with  $h_i = 1$  for all  $i$ ), this map is an isomorphism on each filtered piece (given by an invertible triangular matrix, since  $z_1 \in H^\times$ ). This shows the result for  $n = 0$ . Next, if  $n < 0$ , note that  $A[n] = d^{-n} A[0] \cong d^{-n} H[du]$  is a free rank one right  $H[du]$ -module. Thus, it remains to show that  $d^{-n} A[0] \cong A[0] d^{-n}$  for  $n < 0$ . This can be shown using the first of the equations (2.5) (with  $h = 1$ ), the filtration on  $A[0]$ , and that  $z_1 \in H^\times$ .  $\square$

**2.2. The Bernstein-Gelfand-Gelfand Category.** The goal of this subsection is to introduce and develop basic properties of an important category of weight modules of triangular GWAs – an analogue of the Bernstein-Gelfand-Gelfand category  $\mathcal{O}$  [5]. In light of Lemma 2.1 and Theorem 2.2, we make the following assumptions.

**Assumption 2.7.** For the remainder of this paper, assume that  $A = \mathcal{W}(H, \theta, z_0, z_1)$  is a triangular GWA for which  $\hat{H}^{free}$  is non-empty.

These assumptions are satisfied by many of the examples in the literature when  $\theta$  is an algebra automorphism of  $H$ . See [28, Section 8] for the two large “classical” and “quantum” families of examples. Also note here that  $\theta$  is necessarily not of finite order.

We now define and study Category  $\mathcal{O}$  via a series of results that are required in future sections. We omit the proofs as these results are shown in [28] in greater generality. We begin by setting some notation.

**Definition 2.8.** Define the *Verma module* with highest weight  $\lambda \in \hat{H}$  to be  $M(\lambda) := A/(Au + A \cdot \ker \lambda)$ .

**Remark 2.9.** We now list standard properties of Verma modules and  $\mathcal{O}$ ; see [28] for the proofs.

- (1) Given  $n \geq 0$ ,  $\lambda \in \hat{H}$ , and an  $A$ -module  $M$ , we have:  $u^n M_\lambda \subset M_{n*\lambda}$  and  $d^n M_\lambda \subset M_{(-n)*\lambda}$ .
- (2) For all  $\lambda \in \hat{H}^{free}$ ,  $M(\lambda) \in \mathcal{O}$ . It is a weight module with all nonzero weight spaces of weight  $n * \lambda$  for some  $n \leq 0$ .
- (3)  $M(\lambda)$  is generated by a weight vector  $m_\lambda$ , as a free  $\mathbb{F}[d]$ -module. The weight space  $M_{(-n)*\lambda}$  is spanned by  $d^n m_\lambda$ , and  $u m_\lambda = 0$ . From the point of view of  $A$  being a generalized Weyl



algebra (Lemma 2.1),  $M(\lambda)$  is killed by  $\ker_{H[du]}(\lambda)$ , where every  $\lambda \in \widehat{H}$  extends to an algebra map  $\lambda : H[du] \rightarrow \mathbb{F}$  that sends  $du$  to 0.

- (4) For  $\lambda \in \widehat{H}^{free}$ ,  $M(\lambda)$  has a unique simple quotient  $L(\lambda)$ . The modules  $L(\lambda)$  are pairwise non-isomorphic.
- (5)  $A$  has an anti-involution  $i : A \rightarrow A$ , which sends  $d \leftrightarrow u$  and fixes all of  $H$ .
- (6) The anti-involution  $i$  induces a contravariant, involutive, additive “restricted duality functor”  $F$ , which preserves the category  $\mathcal{O}_{\mathbb{N}}$  of finite length objects in  $\mathcal{O}$ . It is defined as follows: given an  $H$ -weight module  $M := \bigoplus_{\mu \in \widehat{H}} M_{\mu}$ , we define  $F(M) := \bigoplus_{\mu \in \widehat{H}} M_{\mu}^*$ . This is an  $A$ -module action via:  $(am^*)(m) := m^*(i(a)m)$  for  $a \in A, m \in M, m^* \in F(M)$ . Then,

$$F(L(\lambda)) = L(\lambda) \quad \forall \lambda \in \widehat{H}^{free}, \quad F^2(M) \cong M \quad \forall M \in \mathcal{O}_{\mathbb{N}}.$$

- (7) Moreover,  $F$  is an exact functor on  $\mathcal{O}_{\mathbb{N}}$ .

We now discuss the structure of Category  $\mathcal{O}$ , which turns out to be somewhat different from the well-studied case of Lie algebras with triangular decomposition [34].

**Proposition 2.10.** *Every module  $M \in \mathcal{O}$  is a direct sum of summands:*

$$M = \bigoplus_{\langle \mu \rangle \in \widehat{H}/\mathbb{Z}} M\langle \mu \rangle,$$

where given  $\mu \in \widehat{H}$ ,  $\langle \mu \rangle := \mathbb{Z} * \mu$  and  $M\langle \mu \rangle := \bigoplus_{n \in \mathbb{Z}} M_{n * \mu}$ . Thus  $M$  has a finite filtration, each of whose subquotients is either a quotient of a Verma module, or else a finite-dimensional weight module  $N$  such that  $\text{wt } N \subset \widehat{H} \setminus \widehat{H}^{free}$ . In particular,  $\mathcal{O}$  is finite length if and only if every Verma module has finite length.

The next result is very useful in determining the structure of modules in  $\mathcal{O}$ .

**Proposition 2.11.** *Given  $\lambda, \mu \in \widehat{H}^{free}$ ,  $M \in \text{Ext}_{\mathcal{O}}^1(L(\mu), L(\lambda))$  is a non-split extension, if and only if there exists  $0 \neq n \in \mathbb{Z}$  such that  $\mu = n * \lambda$  and  $M(\max(\lambda, \mu))$  surjects onto one of  $M$  and  $F(M)$ . In particular, the following are equivalent:*

- (1)  $\text{Ext}_{\mathcal{O}}^1(L(\mu), L(\lambda)) \neq 0$ .
- (2)  $\dim \text{Ext}_{\mathcal{O}}^1(L(\mu), L(\lambda)) = 1$ .
- (3)  $\mathbb{Z} * \lambda = \mathbb{Z} * \mu$  and  $M(\min(\lambda, \mu))$  is the unique maximal submodule in  $M(\max(\lambda, \mu))$ .

In particular,  $[\lambda] = [\mu]$ .

The following result uses Proposition 2.4 to analyze Verma modules in detail in Category  $\mathcal{O}$ , and provides motivation for considering the sets  $[\lambda]$  used in Definition 1.3 and Theorem A.

**Proposition 2.12.** *Fix any triangular GWA  $A$  and  $\mu \in \widehat{H}^{free}$ . Then  $M(\mu)$  is a uniserial module, with unique composition series:*

$$M(\mu) \supset M((-n_1) * \mu) \supset M((-n_2) * \mu) \supset \cdots,$$

where  $0 < n_1 \leq n_2 \leq \cdots$  comprise the set  $\{n \geq 1 : \mu(\tilde{z}_n) = 0\}$ . Thus  $\mathcal{O}$  is finite length if and only if  $[\mu] \cap (-\mathbb{Z}_+ * \mu)$  is finite for every  $\mu \in \widehat{H}^{free}$ . Moreover, the following are equivalent, given  $n \in \mathbb{Z}_+$  and  $\mu \in \widehat{H}^{free}$ :

- (1) The multiplicity  $[M(n * \mu) : L(\mu)]$  is nonzero.
- (2)  $[M(n * \mu) : L(\mu)] = 1$ .
- (3)  $M(\mu) \hookrightarrow M(n * \mu)$ .
- (4)  $(n * \mu)(\tilde{z}_n) = 0$ .
- (5)  $\mu(\tilde{z}_{-n}) = 0$ .

In particular, every submodule of a Verma module is a Verma module. Moreover, the result also provides a “BGG resolution” of every simple object  $L(\mu)$ :  $M(\mu) = L(\mu)$  if  $\dim L(\mu) = \infty$ ; otherwise given  $n_1$  as in Proposition 2.12, we have  $0 \rightarrow M((-n_1) * \mu) \rightarrow M(\mu) \rightarrow L(\mu) \rightarrow 0$ .

**Remark 2.13.** Observe that the restricted dual of every finite-dimensional highest weight module  $M(\mu)/M((-n_r) * \mu)$  (with notation as in Proposition 2.12, and some  $r \geq 0$  such that  $0 < n_r < \infty$ ) is a *lowest weight module*, generated by its lowest weight vector of weight  $(1 - n_1) * \mu$ . In particular for  $r = 1$ , the simple finite-dimensional module  $L(\mu)$  is both a highest weight module and a lowest weight module, akin to semisimple Lie algebras.

**2.3. Category  $\mathcal{O}$ : projectives, blocks, and highest weight categories.** The next step in proving Theorem A is to construct projective modules in  $\mathcal{O}$ . Note from the equivalences in Proposition 2.12 that  $[\lambda]$  is related to partitioning the set of weights  $\widehat{H}^{free}$  (and hence, Category  $\mathcal{O}$ ) into blocks. Thus it is an analogous notion to that of *linkage* for semisimple Lie algebras, as well as to *Condition (S3)* in the axiomatic framework studied in [28]. Note that these three notions coincide when  $A = \mathcal{W}(H, \theta, z_0, z_1) = U(\mathfrak{sl}_2)$ .

We now recall additional standard constructions from [28], for which the following notation is required.

**Definition 2.14.** Set  $A := \mathcal{W}(H, \theta, z_0, z_1)$  as above. Given  $\lambda \in \widehat{H}$  and an integer  $l \geq 0$ , define  $P(\lambda, l) := A/(Au^l + A \cdot \ker \lambda)$ , and  $\mathcal{O}(\lambda, l) \subset \mathcal{O}$  to be the full subcategory of all  $M \in \mathcal{O}$  such that  $u^l M_\lambda = 0$ . We also say that an object  $X$  in  $\mathcal{O}$  has a (*dual*) *Verma flag* if  $X$  has finite filtration in  $\mathcal{O}$  whose subquotients are (restricted duals of) Verma modules.

We now have the following standard results in the study of Category  $\mathcal{O}$ ; we avoid the proofs as these results are shown in greater generality in [28].

- (1)  $\mathcal{O}_{\mathbb{N}}$  is a direct sum of *blocks*:

$$\mathcal{O}_{\mathbb{N}} = \bigoplus_{\mu \in (\widehat{H} \setminus \widehat{H}^{free})/\mathbb{Z}} \mathcal{O}_{\mathbb{N}}(\mathbb{Z} * \mu) \oplus \bigoplus_{[\lambda] \subset \widehat{H}^{free}} \mathcal{O}_{\mathbb{N}}[\lambda]. \quad (2.15)$$

All morphisms and extensions between objects of distinct blocks (i.e., distinct summands) are zero.

- (2) If  $T \subset \widehat{H}$  is finite, then  $\mathcal{O}(T)$  is a finite length, abelian  $\mathbb{F}$ -category.  
(3) For all finite  $T \subset \widehat{H}^{free}$  and all  $\lambda \in \widehat{H}$ , there exists  $l \geq 0$  such that  $\mathcal{O}(T) \subset \mathcal{O}(\lambda, l)$ .  
(4) For all  $l > 0$  and  $\lambda \in \widehat{H}^{free}$ ,  $P(\lambda, l)$  is a projective module in  $\mathcal{O}(\lambda, l)$ .  
(5) Suppose  $\mathcal{O}$  is finite length and  $[\lambda]$  is finite for some  $\lambda \in \widehat{H}^{free}$ . Fix  $l \geq 0$  such that  $\mathcal{O}[\lambda] \subset \mathcal{O}(\lambda, l)$ , and let  $P(\lambda)$  be the direct summand corresponding to the block  $[\lambda]$ , in the decomposition of  $P(\lambda, l)$  according to (2.15). Then:  
  - $P(\lambda)$  is the projective cover of  $L(\lambda)$  in  $\mathcal{O}[\lambda]$  (and hence in  $\mathcal{O}$ ).
  - $P(\lambda)$  has a Verma flag, with Verma subquotients of the form  $M(\mu)$  for  $\mu \in [\lambda]$ .
  - $\mathcal{O}[\lambda]$  has enough projectives and injectives.
(6) If  $\mathcal{O}$  is finite length and  $[\lambda]$  is finite for  $\lambda \in \widehat{H}^{free}$ , then  $\mathcal{O}[\lambda]$  is equivalent to finite-dimensional (left) modules over a finite-dimensional quasi-hereditary algebra  $\mathbf{A}_{[\lambda]}$ . In particular, it is a highest weight category (see [13], as well as [28, Section 3] for further consequences) that satisfies *BGG Reciprocity*:

$$[P(\mu) : M(\nu)] = [M(\nu) : L(\mu)], \quad \forall \mu \in [\lambda], \nu \in \widehat{H}^{free}.$$

The algebra  $\mathbf{A}_{[\lambda]}$  is obtained as follows: if  $[\lambda] = \{\lambda_1 < \lambda_2 < \dots < \lambda_n\} \subset \widehat{H}^{free}$ , and  $P(\lambda_i)$  is the projective cover of  $L(\lambda_i)$  in  $\mathcal{O}[\lambda]$  (and hence in  $\mathcal{O}$ ), then  $\mathbf{A}_{[\lambda]} = \text{End}_{\mathcal{O}}(\bigoplus_{i=1}^n P(\lambda_i)^{\oplus r_i})^{op}$ , for any choice of positive integers  $r_i$ . Up to Morita equivalence, we may choose  $r_i = 1$  for all  $i$ .

We conclude this section by observing that the above standard facts prove the first part of Theorem A except for the algebra  $\mathbf{A}_{[\lambda]}$  being graded Koszul. It is the goal of the following sections to prove the remaining, more involved homological assertions in Theorems A–C.

### 3. PROJECTIVES AND RESOLUTIONS

In this section and the next, we carry out the technical heart of the computations needed to show the main results in this paper. We end the section by proving Theorem B.

The remainder of this paper operates under the following assumptions.

**Assumption 3.1.** Henceforth assume that Assumption 2.7 holds,  $\mathcal{O}$  is finite length, and the block  $[\lambda]$  is finite for some  $\lambda \in \widehat{H}^{free}$ .

We also set some **notation**. Enumerate the weights in the block as follows:  $[\lambda] = \{\lambda_1 < \lambda_2 < \dots < \lambda_n\} \subset \widehat{H}^{free}$ . Given  $1 \leq i, j \leq n$ , we abuse notation and define  $\lambda_i - \lambda_j$  to be the (unique) integer  $n$  such that  $n * \lambda_j = \lambda_i$ . Recall also the notation of  $L_i, M_i, P_i$  as in Equation (1.13).

We begin by ascertaining the structure of every indecomposable projective object  $P(\lambda)$  in  $\mathcal{O}[\lambda]$ .

**Proposition 3.2.** *We work in  $\mathcal{O}[\lambda]$ . For all  $1 \leq i \leq n$ ,  $M_i$  has a finite filtration*

$$M_i \supset M_{i-1} \supset \dots \supset M_1 \supset 0,$$

*with successive subquotients  $L_j$  for  $1 \leq j \leq i$ . Dually, every  $P_i$  has a finite filtration*

$$P_i \supset P_{i+1} \supset \dots \supset P_n \supset 0,$$

*with successive subquotients  $M_j$  for  $i \leq j \leq n$ . Moreover,  $\mathcal{O}[\lambda] \subset \mathcal{O}(\lambda_i, \lambda_n - \lambda_i + 1)$  for all  $i$ .*

*Proof.* The filtration of each Verma module  $M_i$  is discussed in Proposition 2.12. Next,  $\mathcal{O}[\lambda] \subset \mathcal{O}(\lambda_i, \lambda_n - \lambda_i + 1)$  for all  $i$  by [28, Section 3]. Therefore the  $[\lambda]$ -summand of  $P(\lambda_i, \lambda_n - \lambda_i + 1)$  is precisely  $P_i$ , from above. We now consider the structure of  $P(\lambda_i, l)$  for any  $l > 0$  and  $1 \leq i \leq n$ : if  $p_{\lambda_i}$  is the image of 1 in  $P(\lambda_i, n)$ , then

$$0 = N_0 \subset N_1 \subset N_2 \subset \dots \subset N_l = P(\lambda_i, l) = Ap_{\lambda_i},$$

where  $N_k := Au^{l-k}p_{\lambda_i}$ . It is then easy to verify by comparing formal characters that  $N_k = P((l-k) * \lambda_i, k)$ , and that  $N_k/N_{k-1} \cong M((l-k) * \lambda_i) \forall 1 \leq k \leq l$ .

Now set  $l := \lambda_n - \lambda_i + 1$ . Also let  $N_k[\lambda]$  denote the  $[\lambda]$ -component of  $N_k$  under the decomposition (1.6). Then  $N_k[\lambda] = N_{k-1}[\lambda]$  unless that particular subquotient – namely,  $N_k/N_{k-1} = M((l-k) * \lambda_i)$  – equals  $M_j$  for some  $j \geq i$ . Otherwise if  $N_k/N_{k-1} \cong M_j$ , then  $N_k[\lambda]/N_{k-1}[\lambda] = N_k/N_{k-1} \cong M_j$ , by repeated applications of Proposition 2.11 in the abelian category  $\mathcal{O} = \mathcal{O}_{\mathbb{N}}$ .

Since  $P(\lambda_i) = P_i$  is the  $[\lambda]$ -summand of  $P(\lambda_i, \lambda_n - \lambda_i + 1)$  for all  $i$ , we thus obtain the following commuting sequence, by considering only those  $N_k$ 's in  $P(\lambda_i, \lambda_n - \lambda_i + 1)$  which correspond to some  $\lambda_j$  for  $j > i$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \dots \longrightarrow & P_i \\ \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ 0 & \longrightarrow & P(\lambda_n, 1) & \longrightarrow & P(\lambda_{n-1}, \lambda_n - \lambda_{n-1} + 1) & \longrightarrow & \dots \longrightarrow & P(\lambda_i, \lambda_n - \lambda_i + 1) \end{array}$$

Over here, all arrows are inclusions, and the subquotients in the top row are Verma modules  $M_j$  for  $i \leq j \leq n$ . Moreover, each vertical arrow represents the inclusion of the corresponding  $[\lambda]$ -summand, which concludes the proof.  $\square$

**Remark 3.3.** In fact, if  $p_i$  is the image of 1 in  $P(\lambda_i, \lambda_n - \lambda_i + 1)$  (and hence the generator of its quotient  $P_i$  as well), then it is easy to check that  $u^{\lambda_{i+1} - \lambda_i} p_i$  is the image of the generator  $\bar{1}$  in  $P(\lambda_{i+1}, \lambda_n - \lambda_{i+1} + 1)$ . Also note that if we reverse both vertical arrows or the right-hand vertical arrow in any commuting square in the diagram (by the corresponding projection maps onto the  $[\lambda]$ -summands), then we still obtain a commuting square.

The following result provides a projective resolution in  $\mathcal{O}$  of every highest weight module.

**Proposition 3.4.** *Suppose  $0 < s < r \leq n$ . Then the following is a projective resolution of the highest weight module  $M_r/M_s$  in  $\mathcal{O}$ :*

$$0 \rightarrow P_{s+1} \rightarrow P_s \oplus P_{r+1} \rightarrow P_r \rightarrow M_r/M_s \rightarrow 0, \quad (3.5)$$

with the understanding that  $P_{n+1} = 0$ . If  $0 = s < r \leq n$ , then the Verma module  $M_r$  has a projective resolution:

$$0 \rightarrow P_{r+1} \rightarrow P_r \rightarrow M_r \rightarrow 0, \quad \forall 1 \leq r \leq n. \quad (3.6)$$

*Proof.* We begin with the following observation:

$$\mathrm{Hom}_{\mathcal{O}}(P_i, L_j) = \mathrm{Hom}_{\mathcal{O}}(M_i, L_j) = \mathrm{Hom}_{\mathcal{O}}(L_i, L_j) = \delta_{i,j} \mathbb{F}, \quad \forall 1 \leq i, j \leq n. \quad (3.7)$$

Now note first that the theorem holds for all Verma modules  $M_r$  by Proposition 3.2. Thus, for the remainder of the proof we fix  $0 < s < r \leq n$ . Suppose  $0 \rightarrow K \rightarrow P_r \rightarrow M_r/M_s \rightarrow 0$ ; then the kernel  $K$  equals the lift to  $P_r$  of  $M_s \subset M_r = P_r/P_{r+1}$ . In other words,

$$0 \rightarrow P_{r+1} \xrightarrow{\iota} K \xrightarrow{\pi} M_s \rightarrow 0.$$

Next, the surjection  $P_s \twoheadrightarrow M_s$  factors through a map  $P_s \xrightarrow{\varphi} K \xrightarrow{\pi} M_s$ , since  $P_s$  is projective. Thus, define a map  $f : P_{r+1} \oplus P_s \rightarrow K$  via:  $f(m, n) := \varphi(n) - \iota(m)$ . This is a morphism in  $\mathcal{O}[\lambda]$ , and given  $k \in K$ , choose  $n \in P_s$  such that  $\varphi(n) - k \in P_{r+1}$ . Then  $k = f(\varphi(n) - k, n)$ , whence  $f$  is a surjection.

It remains to compute the kernel of  $f$ , which equals  $\{(\varphi(n), n) : \varphi(n) \in P_{r+1}\}$ . Now  $\varphi(n) \in P_{r+1}$  if and only if  $\pi(\varphi(n)) = 0$ , if and only if (from the above factoring of the surjection)  $n$  is in the kernel of  $P_s \twoheadrightarrow M_s$ , if and only if  $n \in P_{s+1}$  (by Proposition 3.2). But then the map  $\psi : P_{s+1} \rightarrow \ker(f)$ , given by  $\psi(n) := (\varphi(n), n)$ , is an  $A$ -module isomorphism, since  $\varphi$  is an  $A$ -module map. This yields the required projective resolution of  $M_r/M_s$  in  $\mathcal{O}$ .  $\square$

An easy consequence of Proposition 3.4 is that we can now compute all Ext-groups between simple objects in the block  $\mathcal{O}[\lambda]$  (and hence, in  $\mathcal{O}$ ), as shown presently. Indeed, the *Hilbert matrix* of the block  $\mathcal{O}[\lambda]$  is defined to be

$$H(E(\mathbf{A}_{[\lambda]}), t)_{i,j} := \sum_{l \geq 0} t^l \dim \mathrm{Ext}_{\mathcal{O}}^l(L_i, L_j). \quad (3.8)$$

The matrix  $H(E(\mathbf{A}_{[\lambda]}), t)$  encodes homological information in the block  $\mathcal{O}[\lambda]$ , and can now be computed explicitly:

**Corollary 3.9.** *For all  $1 \leq i, j \leq n$  and  $l \geq 0$ ,  $\mathrm{Ext}_{\mathcal{O}}^l(L_i, L_j)$  satisfies the formula stated in Theorem B. In particular, the Hilbert matrix of  $\mathcal{O}[\lambda]$  (or of  $E(\mathbf{A}_{[\lambda]})$ ) is the following symmetric tridiagonal  $n \times n$  matrix with determinant 1:*

$$H(E(\mathbf{A}_{[\lambda]}), t) = \begin{pmatrix} 1 & t & 0 & \cdots & 0 & 0 \\ t & 1+t^2 & t & \cdots & 0 & 0 \\ 0 & t & 1+t^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+t^2 & t \\ 0 & 0 & 0 & \cdots & t & 1+t^2 \end{pmatrix}. \quad (3.10)$$

The corollary follows easily from Propositions 2.11 and 3.4; for sake of brevity, we do not elaborate further here, as the steps are similar to those in proving Theorem C(1) below. Note that the determinant of the given  $n \times n$  matrix can be computed by induction on  $n$  and expanding along the last row.

We now prove additional homological properties of the block  $\mathcal{O}[\lambda]$ . Note by Proposition 3.2 that inside each module  $P_j/P_k$  for  $1 \leq j < k \leq n+1$ , sits a copy of the Verma module  $M_{k-1}$ . Thus,

$\dim \operatorname{Hom}_{\mathcal{O}}(M_i, P_j/P_k) \geq 1$  ( $i < k$ ) for  $1 \leq i \leq n$ . We now show that this inequality is actually an equality – namely, that inside each projective cover, there is at most one maximal vector of each possible weight  $\lambda_i$ . In particular, this helps prove one of our main results.

*Proof of Theorem B.* The assertions prior to Equation (1.14) were shown in Proposition 3.2 and Corollary 3.9. We next claim that

$$\dim \operatorname{Hom}_{\mathcal{O}}(P_i, M) = [M : L_i], \quad \forall M \in \mathcal{O}, \quad 1 \leq i \leq n. \quad (3.11)$$

The equation holds because the functions  $\dim \operatorname{Hom}_{\mathcal{O}}(P_i, -)$  and  $[- : L_i]$  are additive on short exact sequences, and both equal  $\delta_{ij}$  when evaluated at a simple object  $L_j$ .

The heart of the proof involves showing the first assertion in Equation (1.14). For this, we first **claim** that  $\dim \operatorname{Hom}_{\mathcal{O}}(M(\mu), P(\lambda, l)) \leq 1$  for all  $\lambda, \mu \in \widehat{H}^{free}$  and all integers  $l \geq 1$ . The claim is obvious if  $\lambda \notin \mathbb{Z} * \mu$ . Now suppose  $\mu = n_0 * \lambda$  for some  $n_0 \in \mathbb{Z}$ , and define  $\max(\lambda, \mu)$  to be  $\mu$  if  $n_0 \geq 0$ , and  $\lambda$  otherwise. Let  $m_\lambda, m_\mu \in \mathbb{Z}_+$  denote the unique integers such that  $m_\lambda * \lambda = m_\mu * \mu = \max(\lambda, \mu)$ ; then  $n_0 = m_\lambda - m_\mu$ . Note that if  $\operatorname{Hom}_{\mathcal{O}}(M(\mu), P(\lambda, l)) \neq 0$  then  $\mu < l * \lambda$ . Now verify that  $P(\lambda, l)_\mu$  is spanned by

$$\{d^{m_\mu+i} u^{m_\lambda+i} \bar{1}_\lambda : 0 \leq i \leq l - 1 - \max(\lambda, \mu)\},$$

where  $\bar{1}_\lambda$  is the generating vector in (the definition of)  $P(\lambda, l)_\lambda$ . (We use here that  $d^{m_\mu+i} u^{m_\lambda+i}$  kills  $\bar{1}_\lambda$  if  $i \geq l - \max(\lambda, \mu)$ .) Use Proposition 2.6 to conclude that  $V := P(\lambda, l)_\mu$  is a finite-dimensional quotient of  $A[m_\lambda - m_\mu]$ , and hence a finite-dimensional  $\mathbb{F}[du]$ -module. It follows that  $\dim \operatorname{coker}(du|_V) = \dim \ker(du|_V) \leq 1$ . Now  $\ker(u|_V) \subset \ker(du|_V)$ , so  $P(\lambda, l)_\mu$  has at most one maximal vector (up to scalar multiples). This proves the claim.

The next step is to note that since  $M_i \hookrightarrow M_n = P_n \hookrightarrow P_j$  for all  $1 \leq j \leq n$ , hence by the claim,

$$1 \leq \dim \operatorname{Hom}_{\mathcal{O}}(M_i, P_j) \leq \dim \operatorname{Hom}_{\mathcal{O}}(M_i, P(\lambda_j, \lambda_n - \lambda_j + 1)) \leq 1.$$

Thus all inequalities are equalities, and  $\dim \operatorname{Hom}_{\mathcal{O}}(M_i, P_j) = 1$  for all  $i, j$ .

We now compute  $\operatorname{Ext}_{\mathcal{O}}^1(M_i, P_j)$ . Note that, given any object  $X$  in  $\mathcal{O}[\lambda]$ , applying the functor  $\operatorname{Hom}_{\mathcal{O}}(-, X)$  to the short exact sequence  $0 \rightarrow P_{i+1} \rightarrow P_i \rightarrow M_i \rightarrow 0$  yields the long exact sequence:

$$0 \rightarrow \operatorname{Hom}_{\mathcal{O}}(M_i, X) \rightarrow \operatorname{Hom}_{\mathcal{O}}(P_i, X) \rightarrow \operatorname{Hom}_{\mathcal{O}}(P_{i+1}, X) \rightarrow \operatorname{Ext}_{\mathcal{O}}^1(M_i, X) \rightarrow \operatorname{Ext}_{\mathcal{O}}^1(P_i, X) \rightarrow \cdots$$

The last term is zero since  $P_i$  is projective. (Also note that all higher Ext-groups are zero.) Thus the Euler characteristic of the terms listed above is zero, which yields via (3.11):

$$\dim \operatorname{Ext}_{\mathcal{O}}^1(M_i, X) = [X : L_{i+1}] - [X : L_i] + \dim \operatorname{Hom}_{\mathcal{O}}(M_i, X), \quad \forall 1 \leq i \leq n, \quad X \in \mathcal{O}[\lambda]. \quad (3.12)$$

Now apply Equation (3.12) for  $X = P_j$ ; then there are two cases. First, if  $1 \leq i < j$ , then  $[P_j : L_i] = [P_j : L_{i+1}]$  by Proposition 3.2, so by Equation (3.12),

$$\dim \operatorname{Ext}_{\mathcal{O}}^1(M_i, P_j) = \dim \operatorname{Hom}_{\mathcal{O}}(M_i, P_j) = 1.$$

Similarly, if  $i \geq j$ , then  $[P_j : L_i] = [P_j : L_{i+1}] + 1$ , whence  $\dim \operatorname{Ext}_{\mathcal{O}}^1(M_i, P_j) = 0$ .

We now show the results for  $\operatorname{Hom}_{\mathcal{O}}(M_i, P_j/P_k)$  and  $\operatorname{Ext}_{\mathcal{O}}^1(M_i, P_j/P_k)$  simultaneously. We assume below that  $k \in (j, n+1)$ , since the  $k = n+1$  case follows from the above analysis. First suppose that  $i \geq k$ , and apply  $\operatorname{Hom}_{\mathcal{O}}(M_i, -)$  to the short exact sequence  $0 \rightarrow P_k \rightarrow P_j \rightarrow P_j/P_k \rightarrow 0$ , to obtain:

$$0 \rightarrow \operatorname{Hom}_{\mathcal{O}}(M_i, P_k) \rightarrow \operatorname{Hom}_{\mathcal{O}}(M_i, P_j) \rightarrow \operatorname{Hom}_{\mathcal{O}}(M_i, P_j/P_k) \rightarrow \operatorname{Ext}_{\mathcal{O}}^1(M_i, P_k) \rightarrow \cdots$$

Since the last term is zero from above, computing the Euler characteristic via the above analysis yields:  $\operatorname{Hom}_{\mathcal{O}}(M_i, P_j/P_k) = 0$ . Now use Equation (3.12) with  $X = P_j/P_k$  to conclude that  $\operatorname{Ext}_{\mathcal{O}}^1(M_i, P_j/P_k) = 0$  for all  $1 \leq j < k \leq i$ .

We next carry out a similar analysis for  $i < k$ , applying  $\text{Hom}_{\mathcal{O}}(M_i, -)$  to the short exact sequence  $0 \rightarrow P_k \rightarrow P_j \rightarrow P_j/P_k \rightarrow 0$ , to obtain:

$$\begin{aligned} 0 \rightarrow \text{Hom}_{\mathcal{O}}(M_i, P_k) \rightarrow \text{Hom}_{\mathcal{O}}(M_i, P_j) \rightarrow \text{Hom}_{\mathcal{O}}(M_i, P_j/P_k) \\ \rightarrow \text{Ext}_{\mathcal{O}}^1(M_i, P_k) \rightarrow \text{Ext}_{\mathcal{O}}^1(M_i, P_j) \rightarrow \cdots \end{aligned} \quad (3.13)$$

There are now two sub-cases:

- (1) First suppose that  $j \leq i < k$ . Since the last term in (3.13) is zero from above, computing the Euler characteristic via the above analysis yields:  $\text{Hom}_{\mathcal{O}}(M_i, P_j/P_k) = 1$ . Now use Equation (3.12) with  $X = P_j/P_k$  to get:  $\text{Ext}_{\mathcal{O}}^1(M_i, P_j/P_k) = 0$  for  $1 \leq j \leq i < k$ .
- (2) If instead  $i < j$ , then first note that  $\text{Hom}_{\mathcal{O}}(M_i, P_j/P_k) \neq 0$ . Additionally, in the long exact sequence (3.13), the first two terms are one-dimensional, whence

$$0 \neq \text{Hom}_{\mathcal{O}}(M_i, P_j/P_k) = \ker(\text{Ext}_{\mathcal{O}}^1(M_i, P_k) \rightarrow \text{Ext}_{\mathcal{O}}^1(M_i, P_j)).$$

Now since  $\text{Ext}_{\mathcal{O}}^1(M_i, P_k)$  is one-dimensional, it follows that so is  $\text{Hom}_{\mathcal{O}}(M_i, P_j/P_k)$ . Finally, apply Equation (3.12) with  $X = P_j/P_k$  to get:  $\dim \text{Ext}_{\mathcal{O}}^1(M_i, P_j/P_k) = 1$  if  $1 \leq i < j < k$ .

Thus we have shown the first assertion in (1.14). The second assertion is clear for  $l \geq 2$ ; in fact,  $\text{Ext}_{\mathcal{O}}^l(P_j/P_k, X) = 0$  for all  $X \in \mathcal{O}[\lambda]$  and  $l \geq 2$ .

It remains to show the second assertion in (1.14) for  $l = 0, 1$ . First note that if  $\varphi : P_j/P_k \rightarrow M_r/M_s$  is nonzero, then the generating vector  $\bar{1}_{\lambda_j} \in P_j/P_k$  maps to a nonzero weight vector in  $M_r/M_s$  of weight  $\lambda_j$ . Therefore  $\lambda_s < \lambda_j \leq \lambda_r$ , i.e.,  $s < j \leq r$ . Moreover, any such nonzero homomorphism is unique since  $\dim(M_r/M_s)_{\lambda_j} \leq 1$ . This shows that  $\dim \text{Hom}_{\mathcal{O}}(P_j/P_k, M_r/M_s) = \mathbf{1}(s < j \leq r)$ , as claimed. Finally, apply  $\text{Hom}_{\mathcal{O}}(-, M_r/M_s)$  to the short exact sequence

$$0 \rightarrow P_k \rightarrow P_j \rightarrow P_j/P_k \rightarrow 0$$

to obtain the long exact sequence:

$$\begin{aligned} 0 \rightarrow \text{Hom}_{\mathcal{O}}(P_j/P_k, M_r/M_s) \rightarrow \text{Hom}_{\mathcal{O}}(P_j, M_r/M_s) \rightarrow \text{Hom}_{\mathcal{O}}(P_k, M_r/M_s) \\ \rightarrow \text{Ext}_{\mathcal{O}}^1(P_j/P_k, M_r/M_s) \rightarrow \text{Ext}_{\mathcal{O}}^1(P_j, M_r/M_s) \rightarrow \cdots \end{aligned}$$

Since the last term is zero, and the Euler characteristic of this terms of the sequence displayed above is zero as well, we compute using Equation (3.11):

$$\begin{aligned} \dim \text{Ext}_{\mathcal{O}}^1(P_j/P_k, M_r/M_s) &= [M_r/M_s : L_k] - [M_r/M_s : L_j] + \mathbf{1}(s < j \leq r) \\ &= \mathbf{1}(s < k \leq r) - \mathbf{1}(s < j \leq r) + \mathbf{1}(s < j \leq r) = \mathbf{1}(s < k \leq r), \end{aligned}$$

as claimed.  $\square$

For completeness, we also compute the morphisms between highest weight modules and quotients of Verma modules.

**Proposition 3.14.** *Fix integers  $0 \leq s < r \leq n$  and  $0 \leq j < k \leq n + 1$ . Then,*

$$\begin{aligned} \dim \text{Hom}_{\mathcal{O}}(M_r/M_s, M_k/M_j) &= \mathbf{1}(s \leq j < r \leq k), & \text{if } k \leq n, \\ \dim \text{Hom}_{\mathcal{O}}(M_r/M_s, P_j/P_k) &= \delta_{s,0} \mathbf{1}(r < k), & \text{if } j \geq 1. \end{aligned} \quad (3.15)$$

*Proof.* We will use the following consequence of Equation (3.11) without further reference:

$$\dim \text{Hom}_{\mathcal{O}}(P_i, M_k/M_j) = [M_k/M_j : L_i] = \mathbf{1}(j < i \leq k). \quad (3.16)$$

We now show the first assertion. If  $s = 0$ , then  $\text{Hom}_{\mathcal{O}}(M_r, M_k/M_j) = \text{Hom}_{\mathcal{O}}(P_r, M_k/M_j)$  has dimension  $\mathbf{1}(j < r \leq k)$ . If instead  $s > 0$ , then  $\text{Hom}_{\mathcal{O}}(M_r/M_s, M_k/M_j)$  consists of precisely the maps  $M_r \rightarrow M_k/M_j$  such that the image of  $M_s$  is killed. By the above analysis, this happens if and only if  $j < r \leq k$  and  $s \leq j$ , proving the first assertion.

Now note that the  $s = 0$  case of the second assertion was shown in Theorem B. If instead  $s > 0$ , then every morphism  $: M_r/M_s \rightarrow P_j/P_k$  gives rise to a morphism  $: M_r \rightarrow P_j/P_k$ . By Theorem B, no such map kills  $M_s \subset M_r$ , so  $\text{Hom}_{\mathcal{O}}(M_r/M_s, P_j/P_k) = 0$  as claimed.  $\square$

#### 4. TILTING MODULES AND SUBMODULES OF PROJECTIVE MODULES

The goal of this section is to prove Theorem C, which classifies all the tilting modules as well as submodules of quotients of projectives  $P_r/P_s$  in the block  $\mathcal{O}[\lambda]$ . A crucial ingredient in this analysis is the study of maps between quotients of projective objects in the block  $\mathcal{O}[\lambda]$ . This is the focus of the next subsection.

**4.1. Graded maps between quotients of projective modules.** Recall that in order to prove Theorem A, we need to study the algebra

$$\mathbf{A}_{[\lambda]} = \text{End}_{\mathcal{O}}(\mathbf{P}_{[\lambda]}), \quad \text{where } \mathbf{P}_{[\lambda]} = \bigoplus_{1 \leq r \leq n} P_r.$$

Our aim in this subsection is to first study the larger algebra

$$\widetilde{\mathbf{A}}_{[\lambda]} = \text{End}_{\mathcal{O}}(\widetilde{\mathbf{P}}_{[\lambda]}), \quad \text{where } \widetilde{\mathbf{P}}_{[\lambda]} = \bigoplus_{1 \leq r < s \leq n+1} P_r/P_s.$$

The first goal is to show that  $\widetilde{\mathbf{A}}_{[\lambda]}$  is a finite-dimensional,  $\mathbb{Z}_+$ -graded  $\mathbb{F}$ -algebra with a distinguished basis, a subset of which spans the subalgebra  $\mathbf{A}_{[\lambda]}$ . We begin by considering one such family of maps.

**Proposition 4.1.** *Given integers  $1 \leq r \leq s \leq n$ , we have the following short exact sequence in the block  $\mathcal{O}[\lambda]$ :*

$$0 \rightarrow P_r/P_s \xrightarrow{f_{r,s}^{++}} P_{r+1}/P_{s+1} \rightarrow F(M_s/M_r) \rightarrow 0, \quad (4.2)$$

where  $F$  is the restricted duality functor defined in Remark 2.9(6).

*Proof.* We begin by proving the claim that there exists an injection  $f_{r,s}^{++} : P_r/P_s \hookrightarrow P_{r+1}/P_{s+1}$ . The proof is by reverse induction on  $r \in [1, s]$ . For  $r = s, s-1$ , the assertion is immediate since  $P_{s-1}/P_s \cong M_{s-1}$  for all  $s$ . Now suppose the assertion holds for  $r+1 \leq s$ . We then have

$$0 \rightarrow P_{r+1}/P_s \rightarrow P_r/P_s \rightarrow M_r \rightarrow 0, \quad (4.3)$$

and  $f_{r+1,s}^{++} : P_{r+1}/P_s \hookrightarrow P_{r+2}/P_{s+1}$ . If we push-forward (4.3) by  $f_{r+1,s}^{++}$  we get an exact sequence

$$0 \rightarrow P_{r+2}/P_{s+1} \rightarrow N \rightarrow M_r \rightarrow 0. \quad (4.4)$$

We now make the sub-claim that the extension  $N$  in (4.4) is the same as the submodule  $N'$  of  $P_{r+1}/P_{s+1}$ , given by:

$$0 \rightarrow P_{r+2}/P_{s+1} \rightarrow N' \rightarrow M_r \rightarrow 0, \quad (4.5)$$

where  $N'$  is the pre-image of  $M_r$  under  $P_{r+1} \twoheadrightarrow M_{r+1}$ . In order to prove the sub-claim, it suffices to prove the following facts:

- (1) The short exact sequences in (4.3) and (4.5) are non-split;
- (2)  $f_{r+1,s}^{++}$  induces an isomorphism  $: \text{Ext}_{\mathcal{O}}^1(M_r, P_{r+1}/P_s) \rightarrow \text{Ext}_{\mathcal{O}}^1(M_r, P_{r+2}/P_{s+1})$ ;
- (3)  $\dim \text{Ext}_{\mathcal{O}}^1(M_r, P_{r+2}/P_{s+1}) = 1$ . (This was already shown in Theorem B.)

We first show (1). By Theorem B, the map  $: \text{Hom}_{\mathcal{O}}(M_r, P_{r+1}/P_s) \rightarrow \text{Hom}_{\mathcal{O}}(M_r, P_r/P_s)$  induced by post-composing with inclusion, is a nonzero map between one-dimensional vector spaces, hence is an isomorphism. It follows that (4.3) is non-split. Similarly, using Theorem B and that  $\text{Hom}_{\mathcal{O}}(M_r, P_{r+2}/P_{s+1}) = \text{Hom}_{\mathcal{O}}(M_r, P_{r+1}/P_{s+1}) = \text{Hom}_{\mathcal{O}}(M_r, N')$ , shows that (4.5) is also non-split.

To show (2), it suffices to show that

$$\mathrm{Hom}_{\mathcal{O}}(M_r, \mathrm{coker}(f_{r+1,s}^{++})) = \mathrm{Ext}_{\mathcal{O}}^1(M_r, \mathrm{coker}(f_{r+1,s}^{++})) = 0,$$

by using an appropriate long exact sequence obtained from the inclusion  $f_{r+1,s}^{++}$ . Now note by Equation (3.11) that the Jordan-Holder factors of  $\mathrm{coker}(f_{r+1,s}^{++})$  are precisely one copy of  $L_j$  for  $r+2 \leq j \leq s+1$ . Thus to show (2) it suffices to prove that  $\mathrm{Hom}_{\mathcal{O}}(M_r, L_j) = 0 = \mathrm{Ext}_{\mathcal{O}}^1(M_r, L_j)$  for  $j \geq r+2$ . But this follows immediately from the projective resolution of  $M_r$ . This concludes the proof of the claim that  $f_{r,s}^{++} : P_r/P_s \rightarrow P_{r+1}/P_{s+1}$  is an injection.

To complete the proof of Equation (4.2), let  $V$  denote the cokernel of the inclusion  $f_{r,s}^{++} : P_r/P_s \hookrightarrow P_{r+1}/P_{s+1}$ . We first show the sub-claim that the vectors

$$\{v_t := u^{\lambda_t - \lambda_{r+1}} 1_{P_{r+1}/P_{s+1}} : r+1 \leq t \leq s\} \quad (4.6)$$

are not contained in  $P_r/P_s$ , and hence have nonzero images in  $V$ . The proof is by contradiction; thus, suppose for some integer  $t \in [r+1, s]$  that  $v_t = u^{\lambda_t - \lambda_{r+1}} 1_{P_{r+1}/P_{s+1}} \in P_r/P_s$ . By the proof of Proposition 3.2,  $Av_t \cong P_t/P_{s+1}$  then embeds into  $P_r/P_s$ . But then Theorem B would imply:

$$1 = \dim \mathrm{Hom}_{\mathcal{O}}(M_s, P_t/P_{s+1}) = \dim \mathrm{Hom}_{\mathcal{O}}(M_s, Av_t) \leq \dim \mathrm{Hom}_{\mathcal{O}}(M_s, P_r/P_s) = 0,$$

which is impossible, and hence shows the sub-claim.

Now consider the module  $V = \mathrm{coker}(f_{r,s}^{++})$ , which is a weight module containing the weight vectors  $v_t$ , and with composition factors  $\{L_t : r+1 \leq t \leq s\}$ . This implies that  $V$  is a finite-dimensional, lowest weight module with specified formal character. Now  $V$  is easily verified to be the dual of the highest weight module  $M_s/M_r$ , which completes the proof.  $\square$

Our next goal is to produce a distinguished  $\mathbb{Z}_+$ -graded basis of  $\widetilde{\mathbf{A}}_{[\lambda]}$ . For this we first introduce the maps

$$f_{jk}^{++} : P_j/P_k \hookrightarrow P_{j+1}/P_{k+1}, \quad f_{jk}^{-\bullet} : P_j/P_k \hookrightarrow P_{j-1}/P_k, \quad f_{jk}^{\bullet-} : P_j/P_k \twoheadrightarrow P_j/P_{k-1}. \quad (4.7)$$

Here,  $f_{jk}^{++}$  was defined in Proposition 4.1, while the other two maps are canonically induced by the inclusion of  $P_j$  in  $P_{j-1}$  for all  $1 \leq j \leq n$ , from Proposition 3.2. Now define for integers  $1 \leq r < s \leq n+1$ ,  $1 \leq j < k \leq n+1$ , and suitable  $t > 0$ :

$$\varphi_{(r,s),(j,k)}^{(t)} := \underbrace{f_{j+1,k}^{-\bullet} \circ \cdots \circ f_{k-t,k}^{-\bullet}}_{k-j-t} \circ \underbrace{f_{k-t-1,k-1}^{++} \circ \cdots \circ f_{r,r+t}^{++}}_{k-r-t} \circ \underbrace{f_{r,r+t+1}^{\bullet-} \circ \cdots \circ f_{r,s}^{\bullet-}}_{s-r-t}. \quad (4.8)$$

Observe that Equation (4.8) shows the maps  $\varphi_{(r,s),(j,k)}^{(t)}$  to be defined only for  $1 \leq t \leq \min(s-r, k-r, k-j)$ . Our next result shows that the family of maps (4.8) provides the aforementioned graded basis of the algebra  $\widetilde{\mathbf{A}}_{[\lambda]}$ .

**Proposition 4.9.** *Setting as in Theorems A and B.*

- (1) *Fix integers  $1 \leq \{r, s\} \leq j \leq k \leq n+1$ . Then the image of the vector*

$$d^{\lambda_j - \lambda_s} u^{\lambda_j - \lambda_r} 1_{P_r/P_k} \in P_r/P_k$$

*generates the submodule  $P_s/P_{s+k-j}$  of  $P_j/P_k \hookrightarrow P_r/P_k$ .*

- (2) *The maps  $f_{jk}^{++}, f_{jk}^{-\bullet}, f_{jk}^{\bullet-}$  generate the  $\mathbb{F}$ -algebra  $\widetilde{\mathbf{A}}_{[\lambda]} = \mathrm{End}_{\mathcal{O}}(\widetilde{\mathbf{P}}_{[\lambda]})$ . Moreover, the maps*

$$\{\varphi_{(r,s),(j,k)}^{(t)} : 1 \leq r < s \leq n+1, 1 \leq j < k \leq n+1, 1 \leq t \leq \min(s-r, k-r, k-j)\}$$

*form a  $\mathbb{Z}_+$ -graded basis of  $\widetilde{\mathbf{A}}_{[\lambda]}$ . Under this grading on  $\widetilde{\mathbf{A}}_{[\lambda]}$ ,*

$$\deg f_{jk}^{++} = \deg f_{jk}^{-\bullet} = 1, \quad \deg f_{jk}^{\bullet-} = 0, \quad \deg \varphi_{(r,s),(j,k)}^{(t)} = 2(k-t) - r - j. \quad (4.10)$$



Furthermore, if  $1 \leq a < b \leq n+1$ , then for all choices of suitable  $u, t$ ,

$$\varphi_{(j,k),(a,b)}^{(u)} \circ \varphi_{(r,s),(j,k)}^{(t)} = \mathbf{1}(u+t+j-k > 0) \varphi_{(r,s),(a,b)}^{(u+t+j-k)}. \quad (4.11)$$

(3) For all integers  $1 \leq r < s \leq n+1$ , the module  $P_r/P_s$  is indecomposable.

One can also show that  $\dim \widetilde{\mathbf{A}}_{[\lambda]} = \frac{(n+1)^5 - (n+1)^3}{24}$  (although this is not used in the paper).

*Proof.*

(1) Using Proposition 3.2, Remark 3.3, and the previously developed theory of Category  $\mathcal{O}$ , note that if  $\varphi : P_j/P_k \hookrightarrow P_r/P_k$ , with the image of  $\varphi$  denoted by  $V \subset P_r/P_k$ , then  $u^{\lambda_j - \lambda_r} 1_{P_r/P_k} = \varphi(1_{P_j/P_k})$ , where  $1_{P_r/P_k}$  is the image of  $\bar{1} \in P(\lambda_r, \lambda_n - \lambda_r + 1) \in \mathcal{O}$ . Thus we may set  $r = j$  without loss of generality. We can also assume that  $s \leq n$ .

Now let  $M \subset P_j/P_k$  denote the submodule generated by  $d^{\lambda_j - \lambda_s} 1_{P_j/P_k}$ . Clearly  $M \twoheadrightarrow M_s$  under the surjection  $P_j/P_k \twoheadrightarrow P_j/P_{j+1} \cong M_{j+1}$ . Thus  $P_s$  maps onto the cyclic  $A$ -module  $M$  by projectivity, yielding a morphism  $\varphi : P_s \twoheadrightarrow M \subset P_j/P_k$  whose image does not lie in  $P_{j+1}/P_k$  (since the image is  $M_s \neq 0$ ). By the analysis in the proof of Theorem C(1),  $\varphi$  factors through an injective map  $P_s/P_{s+k-j} \hookrightarrow P_j/P_k$ , whose image equals  $M$ .

(2) First suppose  $\varphi : P_r/P_s \rightarrow P_j/P_k$  is a nonzero morphism. Then  $[P_j/P_k : L_r] > 0$  by Equation (3.11), which shows using Proposition 3.2 that  $r < k$ . Now suppose for the remainder of this part that  $r < k$ . Clearly,  $\varphi_{(r,s),(j,k)}^{(t)}$  is an  $A$ -module morphism whose image is contained in  $P_{k-t}/P_k$ . Moreover, the image is not contained in  $P_{k-t-1}/P_k$ , by using the analysis after Equation (4.6). Thus, the maps  $\varphi_{(r,s),(j,k)}^{(t)}$  are linearly independent, which shows that

$$\dim \text{Hom}_{\mathcal{O}}(P_r/P_s, P_j/P_k) \geq \mathbf{1}(r < k) \min(s-r, k-r, k-j). \quad (4.12)$$

We now show that the above maps also span the Hom-space. Indeed, suppose  $\varphi \in \text{Hom}_{\mathcal{O}}(P_r/P_s, P_j/P_k)$ ; then composing with the surjection  $P_r \twoheadrightarrow P_r/P_s$  yields a map in  $\text{Hom}_{\mathcal{O}}(P_r, P_j/P_k)$ . By Equation (3.11), this latter space has dimension  $[P_j/P_k : L_r] = \min(k-r, k-j)$ . Thus, assume for each  $t \in (s-r, \min(k-r, k-j)]$  that  $\varphi_t : P_r/P_{r+t} \rightarrow P_j/P_k$  is a morphism with image in  $P_{k-t}/P_k$ . Repeatedly applying Proposition 4.1 shows that the nonzero submodule  $P_s/P_{r+t}$  embeds into the submodule  $P_{k-t+s-r}/P_k \neq 0$ , but not in  $P_{k-t+s-r+1}/P_k$ , once again using the analysis after Equation (4.6) as well as Remark 3.3. It follows that no linear combination of the  $\varphi_t$  is a map between  $P_r/P_s$  and  $P_j/P_k$ . Thus the maps  $\varphi_{(r,s),(j,k)}^{(t)}$  (with  $1 \leq t \leq \min(s-r, k-r, k-j)$ ) form an  $\mathbb{F}$ -basis of  $\text{Hom}_{\mathcal{O}}(P_r/P_s, P_j/P_k)$  for all  $(r, s), (j, k)$ . Consequently, the maps  $f_{jk}^{++}, f_{jk}^{-\bullet}, f_{jk}^{\bullet-}$  generate  $\widetilde{\mathbf{A}}_{[\lambda]}$ , by Equation (4.8).

Now consider  $\varphi_{(j,k),(a,b)}^{(u)} \circ \varphi_{(r,s),(j,k)}^{(t)}$  for  $1 \leq a < b \leq n+1$  and suitable  $u > 0$ . The image under  $\varphi_{(r,s),(j,k)}^{(t)}$  of the generator  $1_{P_r/P_s}$  lies in

$$P_r/P_{r+t} \hookrightarrow P_{k-t}/P_k \hookrightarrow P_j/P_k,$$

so we now ask where this generator goes under the surjection  $P_j/P_k \twoheadrightarrow P_j/P_{j+u}$  (which is the first factor of the composite map  $\varphi_{(j,k),(a,b)}^{(u)}$ ). By the previous part, the generator of  $1_{P_r/P_{r+t}}$  in  $P_j/P_k$  is precisely

$$d^{\lambda_{k-t} - \lambda_r} u^{\lambda_{k-t} - \lambda_j} 1_{P_j/P_k},$$

so under the surjection  $P_j/P_k \twoheadrightarrow P_j/P_{j+u}$ , this generator goes to

$$d^{\lambda_{k-t} - \lambda_r} u^{\lambda_{k-t} - \lambda_j} 1_{P_j/P_{j+u}}.$$

Once again applying the previous part, this vector generates the submodule  $P_r/P_{r+v} \hookrightarrow P_a/P_b$ , with

$$v := u + t + j - k.$$

Thus, the composite map is nonzero if and only if  $v > 0$ , in which case it sends  $P_r/P_s \rightarrow P_r/P_{r+v} \hookrightarrow P_{b-v}/P_b \hookrightarrow P_a/P_b$ . We now verify that  $v$  is indeed at most  $\min(s-r, b-r, b-a)$ :

$$\begin{aligned} t &\leq \min(s-r, k-r, k-j), \quad u \leq \min(b-a, b-j, k-j) \\ \implies v &= u + (t + j - k) \leq u \leq b-a; \\ v &= t + (u + j - k) \leq t \leq s-r; \\ v &= t + u + j - k \leq k-r + b-j + j - k = b-r. \end{aligned}$$

Thus we have shown that Equation (4.11) holds. The proof concludes by observing that Equations (4.8), (4.10), and (4.11) show that  $\text{End}_{\mathcal{O}}(\widetilde{\mathbf{P}}_{[\lambda]})$  is indeed  $\mathbb{Z}_+$ -graded.

(3) Note that  $\text{End}_{\mathcal{O}}(P_r/P_s)$  is a  $\mathbb{Z}_+$ -graded subalgebra of  $\text{End}_{\mathcal{O}}(\widetilde{\mathbf{P}}_{[\lambda]})$ , from the previous part.

We now claim that the only idempotent is  $\varphi_{(r,s),(r,s)}^{(s-r)} = \text{id}_{P_r/P_s}$ , which would show the result. Indeed, observe by the previous part that  $\varphi_{(r,s),(r,s)}^{(s-r)}$  is the only endomorphism having (graded) degree zero. Now it is standard to verify that if  $0 \neq \sum_{t \geq 0} c_t \varphi_t$  is an idempotent with  $\deg \varphi_t = t$ , then  $c_0 = 1$  and  $c_t = 0$  for  $t > 0$ .  $\square$

**4.2. Tilting objects and their submodules.** The results in the previous subsection enable the analysis of the modules  $P_r/P_s$  and the classification of their submodules, as well as of all tilting modules in the block  $\mathcal{O}[\lambda]$ . The classifications require repeated use of the following result.

**Lemma 4.13.** *Fix  $1 \leq r < k \leq n+1$ , and a vector  $0 \neq x \in P_r/P_k$ . Let  $j \in [r, k)$  and  $s \in [1, j]$  denote the unique integers such that (a)  $x \in (P_j/P_k) \setminus (P_{j+1}/P_k)$ , and (b)  $x \bmod (P_{j+1}/P_k) \in P_j/P_{j+1} \cong M_j$  lies in  $M_s \setminus M_{s-1}$ . Then the submodule generated by  $x$  in  $P_r/P_k$  contains the unique copy of the submodule  $P_s/P_{s+k-j}$  of  $P_j/P_k \hookrightarrow P_r/P_k$ .*

Note that the lemma extends the analysis in the proof of Theorem B (see Equation (4.6) and thereafter).

*Proof.* We first claim that the copy of  $P_s/P_{s+k-j}$  inside  $P_j/P_k$  is unique. The claim follows via a careful analysis of the space  $\text{Hom}_{\mathcal{O}}(P_s/P_{s+k-j}, P_j/P_k)$  and its distinguished graded basis, via Proposition 4.9.

Next,  $(Ax)/[Ax \cap (P_{j+1}/P_k)] \cong M_s$  by Proposition 2.12. It follows by Proposition 4.9(1) that  $v_{j,r,s} + v' \in Ax$ , for some  $v' \in P_{j+1}/P_k$ , where

$$v_{j,r,s} := d^{\lambda_j - \lambda_s} u^{\lambda_j - \lambda_r} 1_{P_r/P_k}.$$

Since all objects in the block  $\mathcal{O}[\lambda]$  are weight modules, we may further assume that  $v'$  is a weight vector of weight  $\lambda_s$  (as is  $v_{j,r,s}$ ). By Proposition 4.9(1) again,  $v + 1_{P_s/P_{s+k-j}} \in Ax$  for some weight vector  $v \in P_{j+1}/P_k$  of weight  $\lambda_s$ . Now observe that if  $v \in (P_l/P_k) \setminus (P_{l+1}/P_k)$  for some  $l < k$ , then  $v$  is a weight vector of weight  $\lambda_s$  under the quotient map  $P_l/P_k \twoheadrightarrow M_l$ . It follows that  $u \cdot v \in P_{l+1}/P_k$ . Applying  $u$  repeatedly, one observes that  $u^{\lambda_{k-j+s-1} - \lambda_s}$  kills  $v$ , and sends  $1_{P_s/P_{s+k-j}}$  via Proposition 4.9(1) to the generating highest weight vector in the Verma submodule  $M_{k-j+s-1} \subset P_s/P_{s+k-j}$ . It follows that  $M_{k-j+s-1} \subset Ax$ .

By a similar argument,  $u^{\lambda_{k-j+s-2} - \lambda_s}$  sends  $v$  to a weight vector in  $M_{k-j+s-1}$  and  $1_{P_s/P_{s+k-j}}$  to a generator of  $P_{s+k-j-2}/P_{s+k-j}$ . By the previous paragraph, it follows that  $P_{s+k-j-2}/P_{s+k-j} \subset Ax$ . Proceeding inductively along these lines, we conclude that  $P_s/P_{s+k-j} \subset Ax$ .  $\square$

We now prove another of our main theorems in the present paper.

*Proof of Theorem C.*

(1) First observe that if  $s = k = n + 1$ , then using Proposition 4.9 and Equation (3.11):

$$\dim \operatorname{Hom}_{\mathcal{O}}(P_r, P_j) = [P_j : L_r] = \min(n+1-r, n+1-j) = n+1 + \min(-r, -j) = n+1 - \max(r, j). \quad (4.14)$$

Next, apply  $\operatorname{Hom}_{\mathcal{O}}(-, P_j/P_k)$  to the short exact sequence  $0 \rightarrow P_s \rightarrow P_r \rightarrow P_r/P_s \rightarrow 0$ , and note that the Euler characteristic of the corresponding long exact sequence is zero. Thus, we compute using the above analysis:

$$\begin{aligned} & \dim \operatorname{Ext}_{\mathcal{O}}^1(P_r/P_s, P_j/P_k) \\ &= \dim \operatorname{Hom}_{\mathcal{O}}(P_s, P_j/P_k) - \dim \operatorname{Hom}_{\mathcal{O}}(P_r, P_j/P_k) + \dim \operatorname{Hom}_{\mathcal{O}}(P_r/P_s, P_j/P_k) \\ &= [P_j/P_k : L_s] - [P_j/P_k : L_r] + \mathbf{1}(r < k) \min(s-r, k-r, k-j) \\ &= \max(k, s) - \max(j, s) + \max(j, r) - \max(k, r) + \mathbf{1}(r < k) \min(s-r, k-r, k-j). \end{aligned} \quad (4.15)$$

We now explicitly compute the last expression in (4.15), and verify that it equals precisely  $\mathbf{1}(r \leq j)\mathbf{1}(s \leq k)(\min(0, j-s) + \min(s-r, k-j))$ . There are three possible cases, and in each of them the verification is straightforward:

- (a)  $r \leq j, s \leq k$ : In this case, the last expression in (4.15) equals
$$k - \max(j, s) + j - k + \min(s-r, k-r, k-j) = \min(0, j-s) + \min(s-r, k-j).$$
- (b)  $j < r < k$ : In this case, the last expression in (4.15) equals
$$\max(k, s) - s + r - k + \min(s-r, k-r) = -\min(k, s) + r + \min(s, k) - r = 0.$$
- (c)  $r \notin (j, k]; s > k$ : In this case, the last expression in (4.15) equals

$$s - s + \max(j, r) - \max(k, r) + \mathbf{1}(r \leq j) \min(s-r, k-r, k-j),$$

and this is easily verified to equal 0 in the two sub-cases:  $r \leq j$  and  $r > k$ .

The above three cases prove the second of the Equations (1.16). Finally, to compute the higher Ext-groups, a similar computation to the ones above, using  $\operatorname{Hom}_{\mathcal{O}}(-, X)$  for any  $X \in \mathcal{O}[\lambda]$ , reveals that  $\operatorname{Ext}_{\mathcal{O}}^l(P_j/P_k, X) = 0$  for  $X \in \mathcal{O}[\lambda]$  and  $l \geq 2$ .

(2) The first observation is that every submodule  $N \subset P_r/P_s$  has a filtration:

$$0 \subset N \cap (P_{s-1}/P_s) \subset N \cap (P_{s-2}/P_s) \subset \cdots \subset N \cap (P_r/P_s) = N,$$

whose subquotients are submodules of Verma modules  $(P_j/P_s)/(P_{j+1}/P_s) \cong M_j$ , hence are Verma modules themselves. It follows that every submodule  $N \subset P_r/P_s$  has a Verma flag.

Next, if  $N \neq 0$ , then  $N \cap (P_{s-1}/P_s)$  is a nonzero submodule by Lemma 4.13 and Proposition 4.9, so it necessarily contains the submodule  $L_1 \subset M_{s-1} = P_{s-1}/P_s \subset P_r/P_s$ . It follows that every submodule of  $P_r/P_s$  is indecomposable.

Finally, we produce a bijection between the set of submodules  $N \subset P_r/P_s$  and the decreasing sequences specified in the statement of the result. Given a decreasing sequence  $s-1 \geq m_l > \cdots > m_1 \geq 1$ , construct the corresponding module  $N \subset P_r/P_s$  as follows: first set  $N_s = 0$ . Now given  $N_j$  for some  $s-l < j \leq s$ , define  $N_{j-1}$  to be the lift to  $N_j$  of the Verma submodule  $M_{m_j+l-s-1} \subset M_{j-1} \cong (P_{j-1}/P_s)/(P_j/P_s)$ . In other words,

$$0 \rightarrow N_j \rightarrow N_{j-1} \rightarrow M_{m_j+l-s-1} \rightarrow 0.$$

In constructing  $N_{j-1}$ , we crucially use Lemma 4.13, since the lift of  $M_{m_j+l-s-1}$  to  $P_r/P_s$  equals  $P_{m_j+l-s-1}/P_{m_j+l-s-1+s-r}$ , and since  $\ker(P_{m_j+l-s-1}/P_{m_j+l-s-1+s-r} \twoheadrightarrow M_{m_j+l-s-1})$  is necessarily contained in  $N_j$  by the hypothesis that the sequence of  $m_j$  is strictly decreasing. Proceeding inductively, we obtain the desired submodule  $N = N_{s-l} \subset P_{s-l}/P_s \subset P_r/P_s$ .

Conversely, given  $N \subset P_r/P_s$ , let  $l$  be the unique integer such that  $N \subset P_{s-l}/P_s$  but  $N \not\subset P_{s-l+1}/P_s$ . Now consider the filtration

$$0 \subset N \cap (P_{s-1}/P_s) \subset N \cap (P_{s-2}/P_s) \subset \cdots \subset N \cap (P_{s-l}/P_s) = N,$$

whose subquotients are submodules of Verma modules  $(P_j/P_s)/(P_{j+1}/P_s) \cong M_j$ , hence are Verma modules themselves. Denote the subquotients as  $M_{m_{s-1}}, \dots, M_{m_{s-l}}$ . Now choose  $1 \leq j \leq l$ , and any weight vector  $n'_j \in N \cap (M_{s-j}/M_s)$  whose image modulo  $N \cap (P_{s-j+1}/P_s)$  is the highest weight vector in  $M_{m_{s-j}}$ . It follows by applying Lemma 4.13 to  $x = n'_j$  that  $s-1 \geq m_{s-1} > m_{s-2} > \dots > m_{s-l} \geq 1$ . Now it is not hard to show that the two maps are inverse assignments, leading to the aforementioned bijection.

In particular, the number of such submodules equals the number of such decreasing subsequences, which can be of lengths  $l = 0, 1, \dots, s-r$ . It follows that there are precisely

$$\sum_{l=0}^{s-r} \binom{s-1}{l} \text{ such sequences.}$$

(3) **Step 1:** We first show the following more general result:

*Given integers  $1 \leq j \leq r \leq s \leq n+1$ , define  $M_{r,s} := P_1/P_{1+s-r}$ . Then  $M_{r,s}$  embeds into  $P_j/P_s$ , and its cokernel has a finite filtration, with subquotients*

$$\begin{aligned} &F(M_{s-r}), F(M_{s-r+1}), \dots, F(M_{s-j-1}), F(M_{s-j}); \\ &F(M_{s-j+1}/M_1), F(M_{s-j+2}/M_2), \dots, F(M_{s-1}/M_{j-1}). \end{aligned}$$

To show the above result, we begin by observing via Proposition 4.1 that

$$M_{r,s} = P_1/P_{1+s-r} \hookrightarrow P_2/P_{2+s-r} \hookrightarrow P_1/P_{2+s-r},$$

and the subquotients are  $F(M_{s-r}/M_1)$  and  $L_1 = F(M_1)$  respectively. Moreover, if we denote by  $v_1$  the generator  $1_{P_1/P_{2+s-r}}$ , then for all  $1 \leq t \leq 1+s-r$ , the vector  $u^{\lambda_t - \lambda_1} v_1$  does not lie in  $M_{r,s}$  by the analysis in Proposition 4.1. It follows by lowest weight theory that the cokernel of the inclusion  $P_1/P_{1+s-r} \hookrightarrow P_1/P_{2+s-r}$  is precisely  $F(M_{s-r})$ . Now change  $s$  to  $s+1, s+2, \dots$  in order to show that the inclusions

$$M_{r,s} = P_1/P_{1+s-r} \hookrightarrow P_1/P_{2+s-r} \hookrightarrow \dots \hookrightarrow P_1/P_{s-j+1}$$

have respective cokernels equal to  $F(M_{s-r}), F(M_{s-r+1}), \dots, F(M_{s-j})$ . This shows the first row of subquotients in the statement above. The second row of subquotients immediately follows by applying Proposition 4.1 to the successive inclusions

$$P_1/P_{s-j+1} \hookrightarrow P_2/P_{s-j+2} \hookrightarrow \dots \hookrightarrow P_j/P_s.$$

**Step 2:** We now conclude the proof. Note from Proposition 4.9 below that  $P_j/P_k$  is indecomposable for all  $1 \leq j \leq k \leq n+1$ . Now apply the previous step with  $j = 1$  and  $s = r = k+1$  for some  $1 \leq k \leq n$ . It follows that  $M_{k,k} = 0 \hookrightarrow P_1/P_{k+1}$ , and the cokernel has a dual Verma flag. Thus  $T_k := P_1/P_{k+1}$  is an indecomposable tilting module. Using Equation (3.11), it is easily verified that

$$[T_k : L_j] > 0 \Rightarrow j \leq k; \quad [T_k : L_k] = 1, \quad \forall 1 \leq k \leq n. \quad (4.16)$$

It follows by [16, Theorem A4.2(i)] that every indecomposable tilting module is isomorphic to  $T_k$  for a unique  $1 \leq k \leq n$ . Next,  $F(T_k)$  is also an indecomposable tilting module, so  $F(T_k) \cong T_k$  by (4.16). Finally, observe from Equation (4.8) that there is a unique  $t = s-r$  such that  $\varphi = \varphi_{(1,k),(1,n+1)}^{(t)} : T_{k-1} \hookrightarrow T_n$  is injective. Moreover, every other map  $\varphi_{(1,k),(1,n+1)}^{(t)}$  has image contained inside  $\text{im}(\varphi)$ . Now dualize the short exact sequence  $0 \rightarrow P_k \rightarrow T_n \rightarrow T_{k-1} \rightarrow 0$  to obtain

$$0 \rightarrow T_{k-1} \rightarrow T_n \rightarrow F(P_k) \rightarrow 0,$$

which concludes the proof by generalities in the highest weight category  $\mathcal{O}[\lambda]$ . □

This classification of tilting modules sets the stage for employing the comprehensive machinery developed by Ringel. We refer the reader to [16, Section A4] for some of the consequences for (tilting theory in) the block  $\mathcal{O}[\lambda]$ . Here we present one application of Theorem C.

**Corollary 4.17.** *The tilting modules form an increasing chain:  $T_1 = L_1 \subset T_2 \subset \cdots \subset T_n = P_1$ ; dually,  $T_n \twoheadrightarrow T_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow T_1$ . Moreover, the following are equivalent for a module  $M$  in  $\mathcal{O}[\lambda]$ :*

- (1)  *$M$  is a submodule of a tilting module  $T_k$  for some  $1 \leq k \leq n$ .*
- (2)  *$F(M)$  is a quotient of  $T_k$ .*

*In this case,  $F(T_k/M)$  is a submodule of  $T_k$ , hence indecomposable with a Verma flag.*

## 5. KOSZULITY AND THE SKL CONDITION

In this section we show that the endomorphism algebra  $\mathbf{A}_{[\lambda]}$  is Koszul and satisfies the Strong Kazhdan-Lusztig condition. The first step is to use the analysis in the preceding sections to prove our remaining main result.

*Proof of Theorem A.* All assertion in the first part, except for the grading, Koszulity, and dimension of  $\mathbf{A}_{[\lambda]}$  follow from the analysis in Section 2.3. Now note that  $\mathbf{A}_{[\lambda]} \subset \text{End}_{\mathcal{O}}(\widetilde{\mathbf{P}}_{[\lambda]})^{op}$ , as discussed in Proposition 4.9. Thus  $\mathbf{A}_{[\lambda]}$  inherits the grading in Proposition 4.9. In particular, the  $\mathbb{Z}_+$ -graded vector space  $V_{ij} := \text{Hom}_{\mathcal{O}}(P_i, P_j)$  has an  $\mathbb{F}$ -basis of maps

$$\varphi_{(i,n+1),(j,n+1)}^{(u)}, \quad 1 \leq u \leq \min(n+1-i, n+1-j) = n+1 - \max(i, j).$$

The grading here is given by  $\deg \varphi_{(i,n+1),(j,n+1)}^{(u)} = 2(n+1-u) - i - j$ . Now define the Hilbert matrix of  $\mathbf{A}_{[\lambda]}$  to be  $H(\mathbf{A}_{[\lambda]}, t)_{i,j} := \sum_{l \geq 0} t^l \dim V_{ij}[l]$  (with respect to this grading). It follows by the above analysis and Corollary 3.9 that

$$H(\mathbf{A}_{[\lambda]}, t)_{ij} = \sum_{u=\max(i,j)}^n t^{2u-i-j} = (H(E(\mathbf{A}_{[\lambda]}), -t)^{-1})_{ij}.$$

Moreover, the algebra  $\mathbf{A}_{[\lambda]}$  has zero-degree graded component equal to the  $\mathbb{F}$ -span of  $\varphi_{(j,n+1),(j,n+1)}^{(n+1-j)} = \text{id}_{P_j}$ , for all  $1 \leq j \leq n$ . Thus,  $\mathbf{A}_{[\lambda]}[0]$  is a semisimple algebra that contains a copy of the unit in  $\mathbf{A}_{[\lambda]}$ :  $1_{\mathbf{A}_{[\lambda]}} = \sum_{i=1}^n \text{id}_{P_i}$ . Using the numerical criterion for Koszulity from [2, Theorem 2.11.1], it follows that  $\mathbf{A}_{[\lambda]}$  is Koszul. This shows the first part of the theorem. (It also follows by [2, Section 2.5] that  $\mathbf{A}_{[\lambda]}$  is the associated graded algebra of its radical filtration.)

For the second part, first define the quiver algebra with relations  $Q_\lambda$  to be the quotient of the path algebra of the double of the  $A_n$ -quiver, modulo the relations (1.10). Now note that the Ext-quiver of  $\mathbf{A}_{[\lambda]}$  is as claimed, by Corollary 3.9. Next, define  $\gamma_i, \delta_i \in \mathbf{A}_{[\lambda]}$ , and the idempotent zero-length path  $e_i$  at  $[i]$ , as follows:

$$e_i \leftrightarrow \varphi_{(i,n+1),(i,n+1)}^{(n+1-i)}, \quad \gamma_i \leftrightarrow \varphi_{(i+1,n+1),(i,n+1)}^{(n-i)} = f_{i+1,n+1}^{-\bullet}, \quad \delta_i \leftrightarrow \varphi_{(i,n+1),(i+1,n+1)}^{(n-i)} = f_{i,n+1}^{++}.$$

Using the explicit relations (4.11) satisfied by the maps  $\varphi_{(r,s),(j,k)}^{(t)}$ , it follows that  $Q_\lambda \twoheadrightarrow \mathbf{A}_{[\lambda]}^{op}$  as  $\mathbb{Z}_+$ -graded algebras. On the other hand, it is easily verified that  $\dim Q_\lambda = 1^2 + \cdots + n^2 = \dim \mathbf{A}_{[\lambda]}^{op}$ . This concludes the proof.  $\square$

**Remark 5.1.** The assumption that  $z_1 \in H^\times$  was required in order to equip  $\mathcal{W}(H, \theta, z_0, z_1)$  with a GWA structure. Thus, algebras defined by (1.2) with  $z_1 \in H \setminus H^\times$  are algebras with triangular decomposition that are not GWAs, by Lemma 2.1.

We now remark that our main results, Theorems A–C, in fact hold for this more general family of algebras (given that  $H$  is commutative, whence  $ud, du$  are transcendental over  $H$ ). The proof in this general setting involves certain explicit computations, which do not require that  $z_1 \in H^\times$ ;

instead, it suffices to assume the weaker condition that  $\mu(z_1) \neq 0 \forall \lambda_1 \leq \mu \leq \lambda_n$ . Specifically, these explicit computations occur only in Section 2, and in proving Proposition 4.9 and Theorem B; the remaining proofs go through unchanged.

Note that in several prominent examples in the literature listed above,  $\widehat{H}^{free} = \widehat{H}$  (see [28, Section 8] for more details). Thus for our results to hold in all blocks of  $\mathcal{O}$  in such examples, we would need to assume that  $z_1$  does not belong to any maximal ideal  $\ker(\lambda)$  for  $\lambda \in \widehat{H}$ . In other words,  $z_1 \in H$  would need to be a unit, which explains the assumption in the present paper.

**Remark 5.2.** As discussed after the statement of Theorem A, the algebra  $\mathbf{A}_{[\lambda]}$  only depends on  $n = |[\lambda]|$ . Thus we define  $\mathcal{A}_n := \mathbf{A}_{[\lambda]}$ . For completeness, we briefly discuss other settings in the literature where the family of algebras  $\mathcal{A}_n$  is studied. Note that  $\mathcal{A}_n$  is the endomorphism algebra of the projective generator of various (singular) blocks of Category  $\mathcal{O}$  over complex simple Lie algebras of low rank; see [18, Sections 6 and 7] and [40, Section 5] for more details. The algebra also features in the study of Category  $\mathcal{O}$  over the Virasoro algebra, in finite blocks, or finite quotients or truncations of thin blocks; this is discussed at length in [6]. Furthermore, the algebra  $\mathcal{A}_n$  and its quadratic dual play a role in the study of hyperplane arrangements, the hypertoric category  $\mathcal{O}$ , perverse sheaves on  $\mathfrak{gl}_n(\mathbb{C})$ , and Cherednik algebras. We refer the reader to [4, 7, 8] for more details.

We next show that the algebra  $\mathbf{A}_{[\lambda]}$  possesses an additional useful homological property called the *Strong Kazhdan-Lusztig condition*. We begin with an Ext-computation that holds in greater generality, in any highest weight category. Say that an object  $X$  in the block  $\mathcal{O}[\lambda]$  is in  $\mathcal{F}(\Delta)$  (respectively  $\mathcal{F}(\nabla)$ ) if  $X$  has a (dual) Verma flag.

**Theorem 5.3** ([16, Proposition A2.2]). *For all  $X \in \mathcal{F}(\Delta), Y \in \mathcal{F}(\nabla)$ , we have:*

$$\dim \operatorname{Ext}_{\mathcal{O}}^l(X, Y) = \delta_{l,0} \sum_{i=1}^n [X : M_i][Y : F(M_i)].$$

Moreover, an object  $X \in \mathcal{O}[\lambda]$  is in  $\mathcal{F}(\Delta)$ , if and only if  $\operatorname{Ext}_{\mathcal{O}}^1(X, F(M_i)) = 0 \forall 1 \leq i \leq n$ .

We now discuss the Strong Kazhdan-Lusztig condition in the block  $\mathcal{O}[\lambda]$ , where  $[\lambda]$  is finite.

**Definition 5.4** ([14, §2.1]). Given a finite length  $A$ -module  $M$ , define its *radical* and *socle* filtrations by:

$$\begin{aligned} \operatorname{Rad}^0 M &= M, & \operatorname{Rad}^j M &= \operatorname{Rad}(\operatorname{Rad}^{j-1} M), \\ \operatorname{Soc}^0 M &= M, & \operatorname{Soc}^j M / \operatorname{Soc}^{j-1} M &= \operatorname{Soc}(M / \operatorname{Soc}^{j-1} M), \quad j > 0. \end{aligned}$$

The block  $\mathcal{O}[\lambda]$  satisfies the *Strong Kazhdan-Lusztig condition (SKL)* relative to a given function  $\ell : [\lambda] \rightarrow \mathbb{Z}$ , if for all integers  $0 \leq l, i$  and  $1 \leq j, k \leq n$ ,

$$\begin{aligned} \operatorname{Ext}_{\mathcal{O}}^l(M_j, \operatorname{Soc}^i(F(M_k))) &\neq 0 \text{ or } \operatorname{Ext}_{\mathcal{O}}^l(\operatorname{Rad}^i(M_j), F(M_k)) \neq 0 \\ \implies l &\equiv \ell(\lambda_j) - \ell(\lambda_k) + i \pmod{2}. \end{aligned}$$

Kazhdan-Lusztig theories and conditions such as (SKL) are desirable properties to have in a highest weight category (equivalently, for quasi-hereditary algebras). A large program has been developed in the literature by Cline, Parshall, and Scott whereby they discuss how such conditions can be attained, as well as specific consequences of having such a theory at hand. See [13, 14] and the references therein for more information on the subject.

We conclude this section by proving the Strong Kazhdan-Lusztig condition for the block  $\mathcal{O}[\lambda]$ .

**Proposition 5.5.** *If Assumption 3.1 holds, then  $\mathcal{O}[\lambda]$  satisfies the Strong Kazhdan-Lusztig condition with respect to the natural length function  $\ell : [\lambda] \rightarrow \mathbb{Z}$  given by  $\ell(\lambda_j) := j = l(M_j)$  for  $1 \leq j \leq n$ .*

In particular,  $\mathbf{A}_{[\lambda]}^!$  is also Koszul (by results in [14]), and  $(\mathbf{A}_{[\lambda]}^!)^\dagger \cong \mathbf{A}_{[\lambda]}$ . Using Theorem 1.9, this provides a second (albeit less direct) proof of the Koszulity of the endomorphism algebra  $\mathbf{A}_{[\lambda]}$ .

*Proof.* By Proposition 2.12,  $\text{Rad}^j M_i = M_{i-j}$  for all  $j$ . Now compute using Theorem 5.3:

$$\dim \text{Ext}_{\mathcal{O}}^l(\text{Rad}^i(M_j), F(M_k)) = \dim \text{Ext}_{\mathcal{O}}^l(M_{j-i}, F(M_k)) = \delta_{l,0} \delta_{j-i,k}.$$

Thus, if  $\text{Ext}_{\mathcal{O}}^l(\text{Rad}^i(M_j), F(M_k))$  is nonzero, then  $l = 0$  and  $j - i = k$ , whence  $\ell(\lambda_j) - \ell(\lambda_k) + i - l = j - k + i - 0 = 2i \equiv 0 \pmod{2}$ , as desired. Now note via the duality functor  $F$  that the socle series of  $F(M_k)$  is dual to the radical series of  $M_k$ , and hence is also uniserial. Thus, the condition involving the socle filtration is verified as above, since  $\text{Soc}^i(F(M_k)) = F(\text{Rad}^i(M_k)) = F(M_{k-i})$ .  $\square$

## 6. THE CATEGORY OF SUB-TRIANGULAR YOUNG TABLEAUX

We now introduce the notion of sub-triangular Young tableaux. This is a hitherto unexplored phenomenon for triangular GWAs, which affords a combinatorial interpretation of morphisms and extensions between distinguished objects of the block  $\mathcal{O}[\lambda]$ .

**6.1. The transfer maps.** We begin with the “transfer map” obtained from Theorem C(2), which sends a submodule  $N \subset P_r/P_s$  to an integer tuple  $(s-1 \geq m_1 > \cdots > m_l \geq 1)$ , for some  $0 \leq l \leq s-r$ . Since  $P_r/P_s \hookrightarrow P_1/P_s \hookrightarrow P_1 = T_n$  via Proposition 4.9(1), we can now define a map  $\Psi$  from a submodule of  $P_1 = T_n$  to tuple of integers, via:

$$N \subset T_n \rightsquigarrow \Psi(N) := (m_1, \dots, m_l). \quad (6.1)$$

Moreover, the integers  $m_j$  are obtained as follows: consider the filtration

$$0 \subset N \cap (P_{s-1}/P_s) \subset N \cap (P_{s-2}/P_s) \subset \cdots \subset N \cap (P_r/P_s) = N.$$

Each subquotient is a submodule of the Verma module  $M_j \cong (P_j/P_s)/(P_{j+1}/P_s)$ , hence is a Verma module. Denote by  $l$  the number of nonzero subquotients, and by  $M_{m_j}$  the subquotient of  $M_{s-j}$  for  $1 \leq j \leq l$ . For instance,  $\Psi(P_r/P_s) = (s-1, s-2, \dots, r)$  for all  $1 \leq r < s \leq n+1$ , which includes all tilting, projective, and Verma modules in the block  $\mathcal{O}[\lambda]$ .

We now explain how to encode the transfer map  $\Psi$  by Young tableaux. First, observe by the above discussion that the submodules  $N \subset T_n$  are in bijection with Young tableaux with consecutively decreasing integer entries in each column (to 1) and each row, and where the topmost cells of each row form the sequence  $\Psi(N)$ . For instance, the following figure corresponds to the submodule  $N_0 := \Psi^{-1}((5, 3, 2))$ .

5		
4	3	2
3	2	1
2	1	
1		

This module is contained in  $P_3/P_6$ , and hence in any  $P_r/P_s$  into which  $P_3/P_6$  embeds. Moreover, as explained in the construction of the map  $\Psi$ , the columns of the diagram correspond to a Verma flag of  $N_0$ , and for each  $j = 1, 2, 3$ , the first  $j$  leftmost columns contain the Jordan-Holder factors in the corresponding submodule of  $N_0$ .

Given any submodule  $N \subset P_r/P_s$ , define  $\mathcal{YT}(N)$  to be the Young tableau with strictly decreasing entries, which is obtained from  $\Psi(N)$  in the above manner. We now observe that the map  $\mathcal{YT}(\cdot)$  behaves well under taking the quotient of one submodule of  $P_r/P_s$  by another. Namely, it is not hard to show that if  $N' \subset N \subset P_r/P_s$ ,  $N/N'$  has a filtration whose subquotients are highest weight modules of the form  $M_{m_j}/M_{m'_j}$ , where  $\Psi(N) = (m_1, \dots, m_l)$  and  $\Psi(N') = (m'_1, \dots, m'_l)$  (with

possibly some zeros added to the end to obtain exactly  $l$  entries). Thus it is natural to define  $\mathcal{YT}(\cdot)$  at any subquotient of  $P_r/P_s$  (and hence of  $T_n$ ), as the skew-tableau

$$\mathcal{YT}(N/N') := \mathcal{YT}(N) \setminus \mathcal{YT}(N'), \quad (6.2)$$

where  $\mathcal{YT}(N')$  embeds into the first few leftmost columns of  $\mathcal{YT}(N)$ , with each cell of  $N'$  mapping to a cell in  $N$  with the same number. Note that such subquotients cover all objects in the block  $\mathcal{O}[\lambda]$  that are generated by a single weight vector.

## 6.2. Dual objects and dual Young tableaux.

We now show that the diagrams of dual objects are closely related – in fact, they are transposes of one another. To examine this in closer detail, first recall that tilting objects are self-dual. This is also reflected in their corresponding Young tableaux, which we now give a name.

**Definition 6.3.** Given an integer  $k \geq 1$ , define  $\mathcal{YT}_k$  to be the labelled triangular diagram:

$k$	$k-1$	$k-2$	$\cdots$	$2$	$1$
$k-1$	$k-2$	$\cdots$	$\cdots$	$1$	
$k-2$	$\vdots$	$\cdots$	$\cdots$		
$\vdots$	$\vdots$	$\cdots$			
$2$	$1$				
$1$					

Observe that  $\mathcal{YT}_k = \mathcal{YT}(T_k)$  for all  $1 \leq k \leq n$ . As is standard, we will denote the conjugate, or transpose, of a Young tableau  $X$  by  $X^T$ . Then  $\mathcal{YT}_k = \mathcal{YT}(T_k)$  is self-conjugate for each  $k$ , corresponding to the self-duality of  $T_k$ . We now show the relation between the Young tableau corresponding to a subquotient of  $T_k$  and its dual, by refining Corollary 4.17.

**Proposition 6.4.** *Suppose  $N \subset T_k = P_1/P_{k+1}$  is a submodule for some  $1 \leq k \leq n$ , and  $\psi := \Psi(N)$ . Then we have the following short exact sequence:*

$$0 \rightarrow N = \Psi^{-1}(\psi) \rightarrow T_k = \Psi^{-1}((k, \dots, 1)) \rightarrow T_k/N = F(\Psi^{-1}(\{k, \dots, 1\} \setminus \psi)) \rightarrow 0, \quad (6.5)$$

*i.e.,  $\Psi(F(T_k/N))$  equals the set  $\{1, \dots, k\} \setminus \Psi(N)$  arranged in decreasing order. In particular, if  $N' \subset N \subset P_r/P_s \subset T_n$  are  $A$ -submodules, then  $N/N', F(N/N')$  are subquotients of  $T_n$ , and*

$$\mathcal{YT}(F(N/N')) = Y(N/N')^T = \mathcal{YT}(N)^T \setminus \mathcal{YT}(N')^T. \quad (6.6)$$

*Moreover, the number of cells labelled  $\boxed{j}$  in  $\mathcal{YT}(N/N')$  is precisely the multiplicity  $[N/N' : L_j]$  for all  $1 \leq j \leq n$ . In particular, the total number of cells in  $\mathcal{YT}(N/N')$  is precisely the length of  $N/N'$ .*

The underlying combinatorial phenomenon is as follows, and verifiable by direct visual inspection: for any submodule  $N \subset T_k$ , the diagram  $\mathcal{YT}_k \setminus \mathcal{YT}(N)$  is the transpose of  $\mathcal{YT}(N_1)$  for a submodule  $N_1 \subset T_k$ . The result says that in fact  $N_1 = F(T_k/N)$ .

*Proof.* We begin by refining the proof of Theorem C(2). In what follows, we use Proposition 4.9(1) and Lemma 4.13 without further reference. Suppose  $\Psi(N) = (m_1, \dots, m_l)$ . We claim that  $N \hookrightarrow P_1/P_{m_1+1} \hookrightarrow P_1/P_{k+1}$ . Indeed, let  $X_j$  denote the image of  $P_1/P_{j+1} \hookrightarrow P_{k-j+1}/P_{k+1} \hookrightarrow P_1/P_{k+1} = T_k$  under the map  $\varphi_{(1,j+1),(1,k+1)}^{(k-j)}$ , and let  $x_j$  denote the image of  $1_{P_1/P_{j+1}}$  in the isomorphic copy  $X_j \subset T_k$ . Now observe that  $u^{\lambda_{m_1}-\lambda_1} x_{m_1}$  generates the Verma module  $M_{m_1} \subset N \cap X_{m_1}$ . Similarly,  $d^{\lambda_{m_1-1}-\lambda_{m_2}} u^{\lambda_{m_1-1}-\lambda_1} x_{m_1}$  generates the Verma submodule  $M_{m_2} \subset M_{m_1-1}$  of the module  $(N/M_{m_1}) \cap (X_{m_1}/M_{m_1})$ . Proceeding inductively, the claim that  $N \subset X_{m_1}$  follows.

Now we begin the proof. Note by Corollary 4.17 that  $F(T_k/N)$  is a submodule of  $F(T_k) = T_k$ . We first show by induction on  $|\Psi(N)|$  that  $\Psi(F(T_k/N)) = \{1, \dots, k\} \setminus \Psi(N)$ . The proof is by induction on  $|\Psi(N)|$ . Thus, given  $N \subset T_k$ , note from above that  $N \subset X_{m_1} \subset T_k$ , so by Step (1) of the



proof of Theorem C(3),  $T_k/N$  has a finite filtration with subquotients  $F(M_k), \dots, F(M_{m_1+1})$ , and  $X_{m_1}/N$ . Because  $M_{m_1} \hookrightarrow N \hookrightarrow X_{m_1} \cong P_1/P_{m_1+1}$ , it follows that  $X_{m_1}/N \cong (P_1/P_{m_1})/(N/M_{m_1})$ . Since  $\Psi(N/M_{m_1})$  has strictly smaller length than  $\Psi(N)$  (including in the case when  $|\Psi(N)| = 1$ ), and since we are left to deal with  $N/M_{m_1} \hookrightarrow P_1/P_{m_1} = T_{m_1-1}$ , we are done by induction.

In particular, the above analysis applied to  $F(T_k/N)$  shows Equation (6.5), and also that  $\mathcal{YT}(F(N)) = \mathcal{YT}(N)^T$  for all sub-objects  $N \subset T_n$ . Next, given sub-objects  $N' \subset N \subset T_n$ , let  $C', C$  denote the cokernels of the maps  $N' \hookrightarrow T_n$  and  $N \hookrightarrow T_n$  respectively. Hence by the above analysis and Corollary 4.17,  $\mathcal{YT}(F(C)) = \mathcal{YT}(C)^T$ , and similarly for  $\mathcal{YT}(F(C'))$ . Now  $N/N' \hookrightarrow C' \twoheadrightarrow C$  is exact, so  $F(C) \hookrightarrow F(C') \twoheadrightarrow F(N/N')$  by duality, and  $F(C') \subset F(T_n)$ . Therefore,

$$\begin{aligned} \mathcal{YT}(F(N/N')) &= \mathcal{YT}(F(C')) \setminus \mathcal{YT}(F(C)) = \mathcal{YT}(C')^T \setminus \mathcal{YT}(C)^T \\ &= (\mathcal{YT}_n \setminus \mathcal{YT}(N'))^T \setminus (\mathcal{YT}_n \setminus \mathcal{YT}(N))^T \\ &= \mathcal{YT}(N)^T \setminus \mathcal{YT}(N')^T = \mathcal{YT}(N/N')^T. \end{aligned}$$

Finally, it suffices to prove the assertion about multiplicities for submodules  $N \subset P_1$ . But this follows from the detailed analysis of the Verma flag of  $N$  as described in Theorem C(2) (and earlier in this section).  $\square$

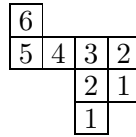
Given this compatibility between dual objects and their associated (dual) Young tableaux, it is natural to ask if these connections can be made precise. In the rest of this section, our goal is to provide a positive answer by introducing a category with such diagrams as objects, and suitable candidates for morphisms. As we will see, we achieve more, by providing combinatorial analogues of the distinguished morphisms  $\varphi_{(r,s),(j,k)}^{(t)}$  (see Equation (4.8) and Proposition 4.9).

**6.3. Objects in the category of sub-triangular Young tableaux.** We now introduce and study a combinatorial category  $\mathcal{Y}_\blacktriangledown$  of sub-triangular Young tableaux, that will contain the aforementioned diagrams corresponding to subquotients of  $T_n$ . In this subsection we analyze the objects of  $\mathcal{Y}_\blacktriangledown$ , and show that they include the diagrams  $\mathcal{YT}(N/N')$  discussed above.

**Definition 6.7.**

- (1) Define a *sub-triangular Young tableau (STYT)* to be a diagram  $X$  that satisfies the following properties:
  - (a)  $\boxed{k} \subset X \subset \mathcal{YT}_k$  for some  $k \geq 1$ .
  - (b)  $X$  is connected.
  - (c) For every row  $R$  and column  $C$  of  $\mathcal{YT}_k$ , the sub-diagrams  $X \cap R$  and  $X \cap C$  are connected.
  - (d) If  $c$  is a cell in  $\mathcal{YT}_k \setminus X$ , then  $X$  cannot contain the cells immediately above  $c$  and to the immediate left of  $c$ , if both cells exist in  $\mathcal{YT}_k$ .
- (2) Given a cell  $c \in X$ , denote the number in it by  $n(c)$ .

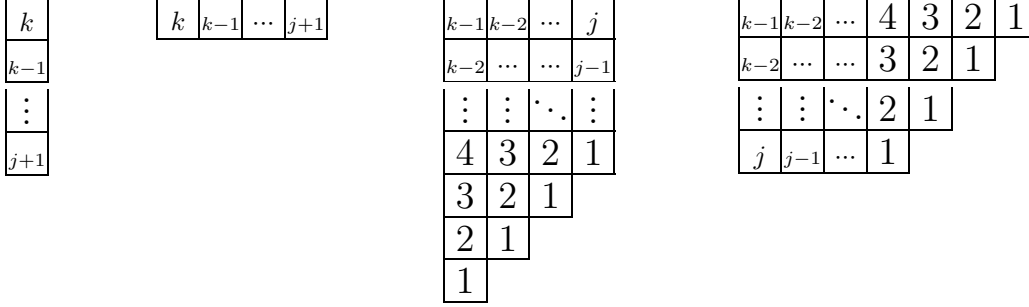
Here is an example of a STYT:



By Proposition 6.4, this diagram equals  $\mathcal{YT}(\Psi^{-1}((6, 4, 3, 2))/\Psi^{-1}((4, 3)))$ .

Henceforth, fix a triangular GWA satisfying Assumption 3.1, and a block  $\mathcal{O}[\lambda]$  with  $[\lambda] = \{\lambda_1 < \dots < \lambda_n \text{ for some } n \geq 1\}$ . Given integers  $0 \leq j \leq k \leq n+1$ , Proposition 6.4 implies that

$\mathcal{YT}(M_k/M_j)$  and  $\mathcal{YT}(F(M_k/M_j))$  (with  $k \neq n$ ), and  $\mathcal{YT}(P_j/P_k)$  and  $\mathcal{YT}(F(P_j/P_k))$  (with  $j \geq 1$ ) are, respectively, the following diagrams, which can all be verified to be STYT's:



**Remark 6.8.** By the analysis in Propositions 4.1 and 6.4, as well as the proof of Theorem C(3), a group of rows at the bottom or a group of columns on the left denotes STYT diagrams of sub-objects in  $\mathcal{O}[\lambda]$ , while a group of rows at the top or a group of columns on the right denotes STYT diagrams of quotients in the block. In order to maintain descending numbers as one moves right or down, the objects in  $\mathcal{F}(\Delta)$  (respectively, in  $\mathcal{F}(\nabla)$ ) are written so that their (dual) Verma subquotients in a (co)standard filtration occur as columns (respectively, rows) of the corresponding STYT's.

**Remark 6.9.** Observe that if we relabelled the families of objects  $\{L_j, M_j, P_j, T_j\}$  under the permutation  $w_o = (j \leftrightarrow n+1-j)$  of  $\{1, \dots, n\}$ , then the STYT diagrams would consist of standard Young tableaux, with strictly increasing (and successive) integers in each row and column. In a parallel representation-theoretic setting involving quantum groups, Young tableaux have connections to crystals; see e.g. [20] and the references therein. In our setting, the cells in a sub-triangular Young tableau correspond not to an  $\mathbb{F}$ -basis for a representation, but to the set of Jordan-Holder factors of the corresponding representation, using Proposition 4.9(1) via the transfer map  $\Psi$ .

We now show that the notion of STYT's is the same as that of  $\mathcal{YT}(\cdot)$ .

**Proposition 6.10.** *For all STYT's  $X \subset \mathcal{YT}_k$ , there exist submodules  $N' \subset N \subset T_k$  such that  $X = \mathcal{YT}(N/N')$ . However, the converse is not necessarily true.*

Note also that to each pair of modules  $N' \subset N \subset T_k$ , there corresponds an integer  $1 \leq l \leq k$ , and two sequences of decreasing integers  $k \geq b_l > b_{l-1} > \cdots > b_1 \geq 1$  and  $b_l > a_l \geq a_{l-1} \geq \cdots \geq a_1 \geq 0$ , such that  $a_j > a_{j-1}$  whenever  $a_{j-1} > 0$ , and  $b_j > a_j$  for all  $j$ . The  $b_j$  form  $\Psi(N)$  and the nonzero  $a_j$  form  $\Psi(N')$ , respectively.

*Proof.* Given an STYT  $X \subset \mathcal{YT}_k$ , it follows easily from the definition of an STYT that the top entries in each column are strictly decreasing, starting at  $k$ . Define  $N$  to be the corresponding submodule of  $T_k$ . Let the last entry in the  $j$ th column be denoted by  $a_j$ ; then if  $a_{j-1} > 0$ , it follows by the definition of an STYT that  $a_j > a_{j-1}$ . Thus, if  $a_1 > a_2 > \cdots > a_r$  denote the entries among the  $a_j$  that are not 1, then Equation (6.2) implies that  $X = \mathcal{YT}(N/N')$ , where  $N' = \Psi^{-1}((a_1 - 1, \dots, a_r - 1))$ . This proves the first assertion; the converse is, however, not true, as is verified from following easy example:  $N = \Psi^{-1}((3, 1))$ ,  $N' = \Psi^{-1}((2))$ .  $\square$

Having shown that the assignment  $\mathcal{YT}(\cdot)$  is compatible with taking subquotients and duals, we now discuss additional properties of  $\mathcal{YT}(\cdot)$  related to generators. We require the following notation.

**Definition 6.11.**

- (1) Given a subset  $X'$  of cells in a STYT  $X$ , the *STYT generated by  $X'$  in  $X$* , denoted by  $\mathcal{YT}(X', X)$ , is the sub-diagram consisting of all cells obtained by traveling from a cell in  $X'$  via a finite sequence of moves, either one cell to the left, or one cell down.

- (2) Given a STYT  $X$ , define its set of *primitive generators* to be any minimal subset of cells  $G_{\min}(X) \subset X$  such that  $X = \mathcal{YT}(G_{\min}(X), X)$ .

**Lemma 6.12.** *The primitive generating set of any STYT  $X$  is unique. If  $X = \mathcal{YT}(N)$  for  $N$  or  $F(N)$  of the form  $M_k/M_j$  or  $P_j/P_k$ , then  $G_{\min}(X)$  is a single cell.*

*Proof.* That every STYT has a minimal generating set is obvious since  $X$  has only finitely many cells; that this set is unique follows by assigning a coordinate to each cell that strictly increases upon moving one cell down or to the left. Now it is clear that each of  $\mathcal{YT}(M_k/M_j)$  and  $\mathcal{YT}(P_j/P_k)$ , or their duals, is generated by one cell, so we are done by the uniqueness of  $G_{\min}(X)$  for all  $X$ .  $\square$

Proposition 4.9(1) and Lemma 4.13 have combinatorial interpretations using STYTs as well. Given any quotient of projective modules  $P_r/P_k$  and a nonzero element  $x \in P_r/P_k$ , first define the *cell of  $x$* , denoted by  $cell(x)$ , as follows: consider  $j \in [r, k)$  such that  $x \in (P_j/P_k) \setminus (P_{j+1}/P_k)$  inside  $P_r/P_k$ . Now consider the submodule generated by  $x$  inside the Verma module  $M_j \cong (P_r/P_k)/(P_{r+1}/P_k)$ , say  $M_s$  for  $s \leq j$ . Then  $cell(x)$  is defined to be the cell numbered  $\boxed{s}$  that is  $j - r$  steps to the left and  $j - s$  steps below the generating cell  $G(\mathcal{YT}(P_r/P_k))$ .

Now if  $N$  is the submodule generated by  $v_{j,r,s} := d^{\lambda_j - \lambda_s} u^{\lambda_j - \lambda_r} 1_{P_r/P_k} \in P_r/P_k$ , then one has:

$$\mathcal{YT}(N) = \mathcal{YT}(A \cdot v_{j,r,s}) = \mathcal{YT}(cell(v_{j,r,s})) \subset \mathcal{YT}(P_r/P_k).$$

The same result holds, albeit with a simpler proof, for the highest weight module  $M_r/M_s$  and for the lowest weight module  $F(M_r/M_s)$ .

**6.4. Morphisms and extensions of sub-triangular Young tableaux.** We now discuss morphisms and extensions between STYTs, as well as the subcategory  $\mathcal{Y}_{\blacktriangledown}$  of decreasing Young tableaux.

**Definition 6.13.** Fix STYTs  $X_1, X_2$ .

- (1) Define a *map*  $\varphi : X_1 \rightarrow X_2$  to be a translation (in the plane) of the diagram  $X_1$ , satisfying the following conditions:
  - (a)  $\varphi$  is *n-equivariant*: for all cells  $c \in X_1$ , either  $\varphi(c)$  is a cell in  $X_2$  with  $n(\varphi(c)) = n(c)$ , or  $\varphi(c)$  is disjoint from  $X_2$ .
  - (b)  $\mathcal{YT}(\varphi(G_{\min}(X_1)), X_2) = X_2 \cap \varphi(\mathcal{YT}(X_1))$  is non-empty.
- (2) A map  $\varphi : X_1 \rightarrow X_2$  is *injective* if  $\varphi(X_1) \subset X_2$ , and *surjective* if  $\varphi(X_1) \supset X_2$ .
- (3) A *morphism*  $X_1 \rightarrow X_2$  is a formal  $\mathbb{F}$ -linear combination of maps  $X_1 \rightarrow X_2$ .
- (4) Define an *extension* of  $X_2$  by  $X_1$  to be a disjoint juxtaposition of  $X_1$  and  $X_2$  (but sharing at least one edge of one cell), such that their (disjoint) union is a STYT, and  $X_1$  is either above or to the right of  $X_2$ .
- (5) Let  $\text{Hom}(X_1, X_2) = \text{Ext}^0(X_1, X_2)$  denote the set of morphisms from  $X_1$  to  $X_2$ , and denote by  $\text{Ext}^1(X_1, X_2)$  the  $\mathbb{F}$ -span of extensions of  $X_2$  by  $X_1$ .
- (6) Define  $\mathcal{YT}(\oplus_{j=1}^k N_j) := \coprod_{j=1}^k \mathcal{YT}(N_j)$  for all objects  $N_j \in \mathcal{O}[\lambda]$  for which  $\mathcal{YT}(N_j)$  is defined.

The following result provides connections between combinatorics and blocks of triangular GWAs. To our knowledge, these connections have not been explored in the literature.

**Theorem 6.14.** *We work in the block  $\mathcal{O}[\lambda]$ , where  $[\lambda] = \{\lambda_1 < \dots < \lambda_n\}$ . Then the assignment  $N \mapsto \mathcal{YT}(N)$  respects morphisms and extensions, including under duality. More precisely, the following analogues of Equations (1.14) and (3.15) hold:*

$$\dim \text{Ext}_{\mathcal{O}}^l(N, N') = \dim \text{Ext}^l(\mathcal{YT}(N), \mathcal{YT}(N')), \quad \forall l = 0, 1, \quad (6.15)$$

if  $N, N'$  satisfy one of the following conditions:

- (1)  $N = P_j/P_k$  as above, and  $N' = M_r/M_s$  as above or  $P_{j'}/P_{k'}$  for some  $1 \leq j' < k' \leq n+1$ ;
- (2)  $N, N'$  are both simple;
- (3)  $N$  is projective and  $N'$  is such that  $\mathcal{YT}(N')$  is defined. If  $N = P_k$  for some  $1 \leq k \leq n$  then  $\dim \text{Hom}(\mathcal{YT}(P_k), \mathcal{YT}(N'))$  equals the multiplicity of the cell  $\boxed{k}$  in  $\mathcal{YT}(N')$ ;

(4)  $N, F(N')$  are Verma modules;

(5) or, if  $(Y, Y')$  are one of the above pairs, and  $N = F(Y'), N' = F(Y)$ . In other words,

$$\dim \text{Ext}^l(\mathcal{YT}(N), \mathcal{YT}(N')) = \dim \text{Ext}^l(\mathcal{YT}(N')^T, \mathcal{YT}(N)^T) = \dim \text{Ext}^l(\mathcal{YT}(F(N')), \mathcal{YT}(F(N))). \quad (6.16)$$

The result follows from the analysis carried out in this paper in the various special cases (and by visual inspection of the corresponding STYTts). In fact the connection in Theorem 6.14 is even stronger. Recall via Proposition 4.9 that the endomorphism algebra of  $\widetilde{\mathbf{P}}_{[\lambda]} := \bigoplus_{1 \leq r < s \leq n+1} P_r/P_s$  is equipped with a  $\mathbb{Z}_+$ -grading, as well as a distinguished basis of morphisms

$$\varphi_{(r,s),(j,k)}^{(t)} : P_r/P_s \rightarrow P_r/P_{r-t} \hookrightarrow P_{k-t}/P_k \hookrightarrow P_j/P_k.$$

It is not hard to verify that the combinatorial counterparts of these morphisms are precisely the distinct maps between the corresponding STYTts:

$$\mathcal{YT}(\varphi_{(r,s),(j,k)}^{(t)}) : \mathcal{YT}(P_r/P_s) \rightarrow \mathcal{YT}(P_r/P_{r+t}) \hookrightarrow \mathcal{YT}(P_{k-t}/P_k) \hookrightarrow \mathcal{YT}(P_j/P_k). \quad (6.17)$$

Moreover, the degree of the map  $\varphi_{(r,s),(j,k)}^{(t)}$  precisely equals the Manhattan distance between the two (unique) generating cells for the STYT map  $\mathcal{YT}(\varphi_{(r,s),(j,k)}^{(t)}) : \mathcal{YT}(P_r/P_s) \rightarrow \mathcal{YT}(P_j/P_k)$ , i.e.,

$$\begin{aligned} \deg \varphi_{(r,s),(j,k)}^{(t)} &= 2(k-t) - r - j \\ &= d_{\text{Manhattan}} \left( G \left( \mathcal{YT}(\varphi_{(r,s),(j,k)}^{(t)})(\mathcal{YT}(P_r/P_s)) \right), G(\mathcal{YT}(P_j/P_k)) \right). \end{aligned} \quad (6.18)$$

This also holds for the maps  $\mathcal{YT}(f_{jk}^{++}), \mathcal{YT}(f_{jk}^{-\bullet}), \mathcal{YT}(f_{jk}^{\bullet-})$  as in (4.10). Thus, we have shown:

**Proposition 6.19.** *The endomorphism algebra of  $\mathcal{YT}(\widetilde{\mathbf{P}}_{[\lambda]}) = \coprod_{1 \leq r < s \leq n+1} \mathcal{YT}(P_r/P_s)$  is naturally isomorphic to  $\text{End}_{\mathcal{O}}(\widetilde{\mathbf{P}}_{[\lambda]})$  as a finite-dimensional  $\mathbb{Z}_+$ -graded  $\mathbb{F}$ -algebra.*

Finally, we define the category of diagrams.

**Definition 6.20.** Denote by  $\mathcal{Y}_{\blacktriangledown}$  the category defined by the following structure:

- (1) The objects are finite disjoint unions of STYTts.
- (2) The morphisms are as in Definition 6.13; thus,  $\mathcal{Y}_{\blacktriangledown}$  is  $\mathbb{F}$ -linear.
- (3) There is a duality functor  $(\cdot)^T : \mathcal{Y}_{\blacktriangledown} \rightarrow \mathcal{Y}_{\blacktriangledown}$  that squares to the identity.
- (4) Extensions between objects are defined as in Definition 6.13.

Notice that every object of  $\mathcal{Y}_{\blacktriangledown}$  is a (possibly disconnected) sub-diagram of  $\mathcal{YT}_k$  for some  $k \geq 1$ . The analysis after Definition 6.13 now shows that triangular GWAs categorify Young diagrams.

**Proposition 6.21.** *Let  $\mathcal{P}$  denote the full subcategory of the block  $\mathcal{O}[\lambda]$  whose objects are  $\{P_r/P_s : 1 \leq r < s \leq n+1\}$ . Then the assignment  $\mathcal{YT}(\cdot)$  is a covariant additive functor from  $\mathcal{P}$  to  $\mathcal{Y}_{\blacktriangledown}$  that respects morphisms and duality.*

As Theorem 6.14 suggests, there are other objects on which the functor  $\mathcal{YT}(\cdot)$  respects additional structure. For instance, morphisms and extensions between Verma modules and dual Verma modules, or between projectives and arbitrary subquotients of tilting modules, are also respected by  $\mathcal{YT}(\cdot)$ . Thus, the discussion in this section naturally leads to the following overarching question, various aspects of which will be considered in future study.

**Question.** Construct a larger category  $\widetilde{\mathcal{Y}}_{\blacktriangledown} \supset \mathcal{Y}_{\blacktriangledown}$  of possibly non-planar diagrams “glued” along edges, and define a functor  $\mathcal{YT}(\cdot) : \mathcal{O}[\lambda] \rightarrow \widetilde{\mathcal{Y}}_{\blacktriangledown}$ , such that the following properties are satisfied:

- (1)  $\mathcal{YT}(\cdot)$  restricts to the functor  $\mathcal{YT}(\cdot)$  studied above, when applied to subquotients of  $T_n = P_1$ .

- (2)  $\widetilde{\mathcal{V}}_{\blacktriangledown}$  is an  $\mathbb{F}$ -linear category, equipped with morphisms, extensions, and a duality functor, which extend to  $\widetilde{\mathcal{V}}_{\blacktriangledown}$  their counterparts in  $\mathcal{V}_{\blacktriangledown}$ .
- (3) The functor  $\mathcal{Y}\mathcal{T}(\cdot)$  is exact, and also respects extensions and duality between objects of  $\mathcal{O}[\lambda]$ . Thus,  $\mathcal{Y}\mathcal{T}(F(N)) = \mathcal{Y}\mathcal{T}(N)^T$ , and Equation (6.15) holds for all  $N, N'$  in  $\mathcal{O}[\lambda]$ .

This question has obvious connections to the representation type of the module category  $\mathcal{O}[\lambda]$  (see e.g. [18] for an analysis in a parallel setting). Note that the exactness of  $\mathcal{Y}\mathcal{T}(\cdot)$  is also natural to expect. For instance, the short exact sequence in Equation (6.5) has a combinatorial counterpart, as does Equation (4.2):

$$\emptyset \rightarrow \mathcal{Y}\mathcal{T}(P_r/P_s) \xrightarrow{\mathcal{Y}\mathcal{T}(f_{r,s}^{++})} \mathcal{Y}\mathcal{T}(P_{r+1}/P_{s+1}) \rightarrow \mathcal{Y}\mathcal{T}(M_s/M_r)^T \rightarrow \emptyset.$$

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